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On liftability of schemes and their Frobenius morphism
PhD dissertation

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I hereby declare that this dissertation is my own work.

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Abstract

The following thesis is concerned with deformation theory over a perfect field of characteristic $p > 0$. We investigate mod p^2 and characteristic 0 liftability of schemes and their Frobenius morphism. In particular, we give an explicit construction of a mod p^2 deformation of a scheme admitting a splitting of the Frobenius morphism. We also provide a few novel examples of schemes which lift neither mod p^2 nor to characteristic 0, but avoid other pathologies of characteristic p geometry. Our constructions are based on interesting line configurations in characteristic p , and rigidity properties of the Frobenius morphism of homogeneous spaces non-isomorphic to the projective space.

Moreover, we study deformations of the Frobenius morphism of singular schemes. We present an explicit criterion for existence of such deformations in case of hypersurface singularities and apply it to canonical singularities of surfaces and ordinary double points in any dimension (answering a question of Bhatt). We also address Frobenius liftability of quotient singularities. In the course of our considerations, we formalise functoriality properties of obstructions classes for lifting schemes and their morphisms. We extensively use the abstract theory of the cotangent complex.

We also provide Macaulay2 scripts for checking mod p^2 and Frobenius liftability.

Keywords: deformation theory, characteristic p geometry, Frobenius morphism, Witt vectors, singularities

AMS Subject Classification: 14G17 positive characteristic ground fields, 14B07 deformation of singularities, 13D10 deformations and infinitesimal methods

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Chapter 1

Introduction

Since the introduction of schemes into the realm of algebraic geometry, varieties defined over fields of positive characteristic p have been source of both trouble and unexpected new possibilities. On the one hand, it is a peculiar property of characteristic p schemes that standard tools like Kodaira vanishing (see [Ray78]) or Hodge decomposition (see [Mum61]) do not necessarily work, and therefore some arguments need to be replaced with a lot of additional work (see, e.g., Enriques' classification of surfaces in [Mum69, BM77, BM76]). On the other hand, every scheme defined over \mathbb{F}_p comes equipped with additional structure of the Frobenius morphism, which can often facilitate otherwise difficult proofs. A remarkable example of this phenomenon is the bend and break technique of Mori (see [Mor79]), which led to the proof of the Hartshorne's conjecture.

The main theme of this dissertation is the *mixed characteristic deformation theory*, which provides a way for characteristic p and characteristic zero algebraic geometry to work in synergy. The key technical methods considered in the field are the procedures of *reduction mod p* and *mixed characteristic lifting*. In this introduction we sketch the basic ideas without any technical details.

1.1 Reduction mod p and characteristic 0 liftability

Let K be a field of characteristic zero. The procedure of reduction mod p of a given variety X/K consists of the following steps. First, we see that since X is of finite type it is locally defined by finitely many equations with coefficients $a_i \in K$. Then we take a scheme \mathcal{X} (called a *spreading out*) defined by the same equations as X but considered over a finitely generated ring $R = \mathbb{Z}[a_1, \dots, a_n] \subset K$. Now, we observe that the scheme \mathcal{X}/R satisfies the following two properties:

- a) it is a *model* of X , that is, the inclusion $\eta: R \rightarrow K$ induces an isomorphism $\mathcal{X}_K \simeq X$,
- b) for every maximal ideal \mathfrak{p} of R the fibre $\mathcal{X}_{\mathfrak{p}} = \mathcal{X} \times_R k_{\mathfrak{p}}$ is defined over a finite field $k_{\mathfrak{p}}$.

Over a dense open subset of $\mathrm{Spec}(R)$, the scheme \mathcal{X} is flat, and therefore the fibres $\mathcal{X}_{\mathfrak{p}}$ reflect the properties of X . We call such fibres reductions mod p of X .

Informally, the mixed characteristic lifting of a scheme Y defined over a field of characteristic p is just the inverse procedure. For simplicity let us assume that $Y \subset \mathbb{P}_k^N$ is projective. First, we see that for any field k of characteristic p there exists a ring S of characteristic zero and a maximal ideal \mathfrak{m} such that $S/\mathfrak{m} \simeq k$. Then we lift the

homogeneous equations of Y from k to S and consider the resulting scheme $\mathcal{Y}/S \subset \mathbb{P}_S^N$, called a *lifting*, whose special fibre satisfies $\mathcal{Y}_m \simeq Y$. However, this time we encounter a certain important caveat. In contrast to the previous situation, there is no simple way to guarantee that the family \mathcal{Y}/S is flat, and therefore it is not clear how to relate the properties of Y and the generic fibre of \mathcal{Y} . In fact, in [Ser61], Serre provides a surprising example (cf. §5.4.1) of a variety defined over a characteristic p field such that no characteristic zero lifting as described above satisfies the flatness requirement.

To emphasize the importance of flatness we present the following example. Let Y be a smooth surface of general type defined over a field k of characteristic p . Suppose that there exists a flat family \mathcal{Y}/S whose special fibre satisfies $\mathcal{Y}_m \simeq Y$. Then the generic fibre \mathcal{Y}_η is a characteristic zero smooth projective surface of general-type. Therefore it satisfies the so-called Bogomolov-Miyaoka-Yau inequality

$$c_1(\mathcal{Y}_\eta)^2 \leq 3c_2(\mathcal{Y}_\eta). \quad (1.1)$$

Using invariance of intersection numbers under flat deformations, we see that the Chern numbers of the special fibre satisfy the same inequality:

$$c_1(\mathcal{Y})^2 \leq 3c_2(\mathcal{Y}).$$

In contrast, in [Eas08] one can find a characteristic p counterexample for the inequality which arises from interesting purely characteristic p line arrangements.

1.2 Liftability mod p^2

It turns out that for many purposes, one does not need to lift a given variety all the way to characteristic zero, and it suffices to have a lifting modulo p^2 . For example, Deligne and Illusie (see [DI87]) showed that for a smooth variety X over a perfect field k of characteristic $p > \dim X$ admitting a lifting over the ring $W_2(k)$ of Witt vectors of length 2, the Hodge–de Rham spectral sequence degenerates (cf. §5.2.1), and the Kodaira vanishing theorem holds. Moreover, under the assumption of mod p^2 liftability, Ogus and Vologodsky (see [OV07]) established the so-called non-abelian Hodge theory, which provides an equivalence between certain categories of decorated vector bundles. Using this result, Langer (see [Lan15]) showed that inequality (1.1) holds for surfaces liftable to $W_2(k)$ (as long as $p > 2$). Counterexamples to Kodaira vanishing in positive characteristic given by Raynaud (see [Ray78]) provide examples of varieties which do not lift to $W_2(k)$. Subsequently, a rational example is given by Lauritzen and Rao (see [LR97]). Another interesting non-liftable varieties appear in the works of Hirokado, Schroöer and Cynk–van Straten [Hir99, Sch04, CvS09]. Those are in fact Calabi–Yau, and thus provide counterexamples to the Bogomolov-Tian-Todorov unobstructedness theorem in characteristic p (in fact, only for $p \leq 3$).

Apart from global mod p^2 liftability, the crucial part of the arguments given in the above mentioned papers is existence of a local lifting of the Frobenius morphism (cf. §5.3.1). For a scheme X together with a $W_2(k)$ -lifting \mathcal{X} , a local Frobenius lifting is determined by an open subset $U \subset X$ and an endomorphism $F': \mathcal{U} \rightarrow \mathcal{U}$ of the induced deformation \mathcal{U} restricting to the Frobenius morphism on $U \subset \mathcal{U}$. In the case of smooth schemes such local liftings always exist because any smooth mod p^2 deformation is étale locally isomorphic with the affine space. On the contrary, singular schemes do not necessarily lift over $W_2(k)$, and those that do, might not admit a compatible lifting of the Frobenius morphism. The importance of local Frobenius liftability is the singular

case was realized in [Bha12, Bha14], where the author generalizes the results of [DI87] and exhibits close relation between local Frobenius liftability and finite-generation of crystalline cohomology of proper singular varieties (see §5.3.2 for the precise statements). The question of existence of Frobenius lifting was also investigated in the global setting. In [BTLM97] it is showed that a smooth projective variety admitting a global Frobenius lifting satisfies certain strong form of the Kodaira-Akizuki-Nakano vanishing theorem, called Bott vanishing (cf. §5.3.1). This in turn leads to a proof of the fact that certain homogeneous spaces do not lift mod p^2 compatibly with the Frobenius morphism.

1.3 Our results

The above considerations suggest the following two basic questions:

Question 1 What are the necessary conditions for a scheme to possess a mod p^2 or characteristic zero lifting?

Question 2 For a given scheme, does there exist a mod p^2 lifting together with a compatible lifting of the Frobenius morphism?

General answers to the questions were given in [Ill71, MS87, Jos07] and stated that above problems can be expressed in a purely cohomological manner. Moreover, in [LS14] the authors investigated the first question in the setting of birational geometry. Our contribution to answering the above questions is the following.

1.3.1 Results concerning mod p^2 and characteristic zero deformations

As far as the first question is concerned, in §6.2 we provide a couple of new examples of varieties which do not admit lifts neither mod p^2 , nor to characteristic zero (some of them do not even lift to any ring A with $pA \neq 0$). However it turns out that they avoid standard characteristic p pathologies. The first construction is given by the blow-up of the two-fold self product of a suitable projective homogeneous space $\neq \mathbb{P}^n$ along the graph of its Frobenius morphism. The easiest examples of such homogeneous spaces are the three-dimensional complete flag variety SL_3/B (isomorphic to the incidence variety $\{x_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2$) and the four-dimensional smooth quadric hypersurface $Q = \{x_0^2 + x_1x_2 + x_3x_4 = 0\} \subset \mathbb{P}^5$ (isomorphic to the Grassmannian $G(2, 4)$ via the Plücker embedding). So the smallest non-liftable examples given by the above construction are of dimension six and eight, with Picard numbers five and three, respectively. The second construction is the variety obtained from \mathbb{P}^3 by blowing up all \mathbb{F}_p -rational points, and then blowing up the strict transforms of all lines connecting \mathbb{F}_p -rational points.

Moreover, in §6.3 we give an alternative proof of the fact that every Frobenius split (cf. §4.4) scheme is $W_2(k)$ -liftable. The advantage of our approach is that we provide the lifting explicitly and functorially with respect to a splitting $\varphi: F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$, in fact as a certain subscheme of the Witt scheme $(X, W_2(\mathcal{O}_X))$.

Finally, in §6.4 we investigate how the property of mod p^2 liftability behaves in families. More precisely, we show, that under mild assumptions suitable for deformation-theoretic considerations (satisfied if e.g. the family is proper or affine), the locus of Witt vector liftable fibres is constructible (see Theorem 6.4.5 for the precise statement).

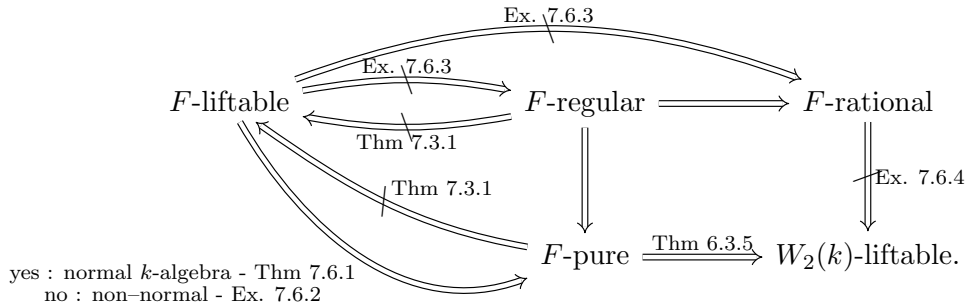
1.3.2 Results concerning liftability of the Frobenius morphism

Concerning the second question, in §7.1 we provide the obstruction theory of the Frobenius morphism in case of singular schemes. Then in §7.2 we express the obstruction classes for lifting Frobenius morphism in an explicit way and derive a computationally feasible criterion for Frobenius liftability of complete intersection affine schemes. Consequently, we apply this criterion in the case of ordinary double points (see §7.3), and thus answer the question asked by Bhatt in [Bha14, Remark 3.14]. As another application of our criterion, in §7.4 we prove that canonical surface singularities are Frobenius liftable. Finally, in §7.5 we present a general approach for proving Frobenius liftability of tame reductions of quotient singularities. The method is based on the spreading out technique combined with functoriality of obstruction classes for lifting morphisms. As a consequence, using classification of tame quotient singularities of surfaces given in [LS14, Proposition 4.2], we give another treatment of Frobenius liftability of canonical singularities of surfaces.

1.3.3 Relation with F -singularities

In the context of the second question, we also relate the notion of Frobenius liftability with standard singularity types defined in characteristic p . In [Kun69] Kunz proved that a scheme X defined in characteristic p is regular if and only if its Frobenius morphism $F: X \rightarrow X$ is flat. This result paved the way for more complex applications of Frobenius in classifications of singularities. Different properties of F , expressed in terms of simple properties of ring morphisms or local dualising complexes, served as motivation for introducing such classes as F -pure, F -split and F -rational singularities (cf. §4.5), which have drawn serious attention of researchers in the previous years (see, e.g., [Smi97, Sch09, Tuc12, ZKT14]).

In §7.6 we present a detailed comparison of the above singularity types and F -liftable singularities. Our results can be summarized by the following diagram which describes all potential relations between the classical F -singularities and the notion considered in this thesis.



The strike-through arrows indicate the existence of counterexamples to the corresponding implications. Moreover, the implication F -liftable \Rightarrow F -pure holds under special assumptions described in the comments below the corresponding arrow.

1.3.4 Additional results concerning deformation theory

As a byproduct of our considerations, in §3.4 we provide a detailed description of functoriality properties of obstruction classes for lifting schemes and their morphisms. To

the best of our knowledge, this results have not been known, or at least do not appear in any of standard references.

1.4 Notation and basic definitions

In this section we introduce the notation and basic definitions concerning Frobenius morphism and deformation theory.

Throughout this thesis we adopt the following conventions. Unless otherwise stated, by k we denote a perfect field of characteristic $p > 0$. All schemes we consider are locally noetherian, quasi-compact and quasi-separated. For every scheme S over k , by $F_S: S \rightarrow S$ (or $F: S \rightarrow S$) we denote the absolute Frobenius morphism of S defined as identity on the level of topological space and by the assignment $f \mapsto f^p$ on the level of structure sheaves. If X is an S -scheme, by $X^{(1/S)}$ (or by $X^{(1)}$ if no confusion can arise) we denote the fibre product $X \times_{S, F_S} S$. This induces the relative Frobenius morphism $F_{X/S}: X \rightarrow X^{(1/S)}$, and the base change morphism $W_{X/S}: X^{(1/S)} \rightarrow X$ described by the diagram

$$\begin{array}{ccccc}
 & & F_X & & \\
 & \nearrow & & \searrow & \\
 X & \xrightarrow{F_{X/S}} & X^{(1/S)} & \xrightarrow{W_{X/S}} & X \\
 & \searrow f & \downarrow f^{(S)} & & \downarrow f \\
 & & S & \xrightarrow{F_S} & S.
 \end{array}$$

Let $X \rightarrow S$ be a morphism of schemes and, let $S \rightarrow S'$ a closed immersion.

Definition 1.4.1. An \tilde{S} -lifting of X/S is a flat morphism $\tilde{X} \rightarrow \tilde{S}$ together with an S -isomorphism $\tilde{X} \times_{\tilde{S}} S \simeq X$.

We say that X/S admits an \tilde{S} -lifting or is \tilde{S} -liftable if there exists a lifting \tilde{X}/\tilde{S} as defined above. In particular, we say that X/k lifts to $W_2(k)$, or is $W_2(k)$ -liftable, or lifts mod p^2 if it admits a lifting $X/\text{Spec } W_2(k)$.

Definition 1.4.2. We say that a scheme X/k lifts *compatibly with Frobenius* or is *Frobenius liftable* (abbr. F -liftable) if there exists a $W_2(k)$ -lifting \tilde{X} of X together with a morphism $\tilde{F}: \tilde{X} \rightarrow \tilde{X}$ over the Frobenius of $W_2(k)$ which restricts to $F_X: X \rightarrow X$, that is, such that the following diagram is commutative:

$$\begin{array}{ccccc}
 & & X & \xrightarrow{F_X} & X \\
 & \swarrow i & \downarrow \tilde{F} & & \swarrow i \\
 \tilde{X} & \xrightarrow{\quad} & \tilde{X} & & \tilde{X} \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \text{Spec}(k) & \xrightarrow{F_k} & \text{Spec}(k) \\
 & \swarrow & \downarrow & & \swarrow \\
 \text{Spec}(W_2(k)) & \xrightarrow{\sigma_2} & \text{Spec}(W_2(k)) & & \text{Spec}(W_2(k))
 \end{array}$$

where $\sigma_2: W_2(k) \rightarrow W_2(k)$ is the Witt vector Frobenius (cf. §2.1.1).

Remark 1.4.3. Alternatively, for a perfect field k , we can reformulate the above definition in terms of the relative Frobenius morphism $F_{X/k}: X \rightarrow X^{(1)}$ and its lifting $\tilde{F}_{X/k}: \tilde{X} \rightarrow \tilde{X} \times_{W_2(k), \sigma} W_2(k)$ over $W_2(k)$.

1.4.1 Notation

1. Let Λ be a complete local ring with residue field k . By $\mathbf{Art}_\Lambda(k)$ we denote the category of Artinian Λ -algebras with residue field k .
2. Let X be a scheme. Throughout by $D(\mathcal{O}_X)$ we denote the derived category of \mathcal{O}_X -modules and by $D_{\mathbf{QCoh}}(X)$ we denote the subcategory of $D(\mathcal{O}_X)$ consisting of complexes of quasi-coherent sheaves. We add $+$ (resp. $-$) in the superscript in order to indicate that we consider the category of bounded below (resp. above) complexes. Moreover, by $D_{\mathbf{Coh}}^\pm(X)$ we denote the full subcategory of $D_{\mathbf{QCoh}}^\pm(X)$ consisting of complexes whose cohomology sheaves are coherent. Under our assumptions, one can prove that any object in $D_{\mathbf{Coh}}^-(X)$ is quasi-isomorphic to a complex of coherent sheaves (see [Huy06, Proposition 3.5]). For every two complexes \mathcal{E}^\bullet and \mathcal{F}^\bullet in $D(\mathcal{O}_X)$ we identify the groups $\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i])$ and $\mathrm{Ext}_{\mathcal{O}_X}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. It is always clear from the context where the complexes are defined and there we usually suppress the subscript indicating the \mathcal{O}_X -linearity condition.

1.5 Structure of the thesis

The thesis is divided into seven chapters including the introduction given in §1. In §2 we present some preliminary results concerning commutative algebra and geometry. Next, in §3 we give a detailed introduction to deformation theory. Subsequent chapters §4 and §5 contain standard tools in characteristic p geometry and classical examples in mixed characteristic deformation theory. In §6 we give our results concerning mod p^2 deformations. Finally, in §7 we present our considerations about liftability of the Frobenius morphism.

1.6 Additional remarks

The thesis is an extended version of author's papers: "Liftability of singularities and their Frobenius morphism modulo p^2 " [Zda16] and "Some elementary examples of non-liftable schemes" [AZ16] (with Piotr Achinger).

Chapter 2

Preliminaries in algebra and geometry

In this chapter we present some preliminary results concerning commutative algebra, homological algebra and geometry. First, in §2.1 we treat the basics of non-equicharacteristic commutative algebra. Then, in §2.2 we recall a few necessary definitions concerning morphisms schemes. Next, in §2.3 we give a brief introduction to derived categories of unbounded complex. Finally, in §2.4 and §2.5 we present the concept of local cohomology and give an exemplary computation for cones over projective schemes.

2.1 Commutative algebra

We begin with a review of basic facts concerning mixed characteristic commutative algebra. First, we give with a brief introduction to Witt vectors of length 2. Next, we recall the local criterion of flatness and derive a few corollaries useful in our future considerations.

2.1.1 Witt vectors

We shall need the following explicit description of Witt vectors of length 2.

Definition 2.1.1 (Witt vectors $W_2(A)$). Suppose A is a commutative ring. We define the ring of Witt vectors $W_2(A)$ to be the set $A \times A$ equipped with addition $+_W$ and multiplication \cdot_W given by the following formulas:

$$(a_0, a_1) +_W (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - P(a_0, b_0)) \quad (2.1)$$

$$(a_0, a_1) \cdot_W (b_0, b_1) = (a_0 b_0, a_0^p b_1 + b_0^p a_1 + p a_1 b_1), \quad (2.2)$$

where $P(a, b)$ is the polynomial $\frac{(a+b)^p - a^p - b^p}{p} \in \mathbb{Z}[a, b]$.

The unit element of $W_2(A)$ is represented by $(1, 0)$ and, in case of \mathbb{F}_p -rings, the prime number p is represented by $(0, 1)$. The natural projection $(a_0, a_1) \mapsto a_0$ gives a ring homomorphism $\pi : W_2(A) \rightarrow A$. In case of characteristic p ring A , the ring $W_2(A)$ possesses a Frobenius endomorphism $\sigma_2 : W_2(A) \rightarrow W_2(A)$ given by the formula $(a_0, a_1) \mapsto (a_0^p, a_1^p)$, which is compatible with the Frobenius endomorphism $F : A \rightarrow A$, i.e., the identity $F \circ \pi = \pi \circ \sigma_2$ holds.

2.1.2 Morphism of Artinian rings

Definition 2.1.2. Any surjective morphism in $\mathbf{Art}_\Lambda(k)$ can be decomposed into a sequence of *small extensions*, i.e., ring surjections $(B, \mathfrak{m}_B) \rightarrow (A, \mathfrak{m}_A)$ with kernel I satisfying $\mathfrak{m}_B I = 0$. Small extensions naturally form a category $\mathbf{SmallExt}(\Lambda, k)$ whose objects are small extensions and morphisms are diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow f_2 & & \downarrow f_1 & & \\ 0 & \longrightarrow & I' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0, \end{array}$$

where f_1, f_2 are rings homomorphisms and ϕ is an induced homomorphism of k -vector spaces.

2.1.3 Flatness criteria

We recall basic facts concerning flatness of modules and algebras. Our presentation is based on [Har10, Section 2].

Lemma 2.1.3 ([Har10, Lemma 2.1]). *Suppose M is a module over a noetherian ring A . Then M is A -flat if and only if for every prime ideal $\mathfrak{p} \subset A$ we have $\mathrm{Tor}^1(M, A/\mathfrak{p}) = 0$.*

As a corollary one obtain the following useful result:

Proposition 2.1.4 ([Har10, Proposition 2.2]). *Let $A' \rightarrow A$ be a surjective homomorphism of noetherian rings whose kernel J has square zero. Then an A' -module M' is flat over A' if and only if:*

1. $M = M' \otimes_{A'} A$ is flat over A , and
2. the natural map $M' \otimes_{A'} J \simeq M \otimes_A J \rightarrow M$ is injective.

Lemma 2.1.5. *Suppose $(A', \mathfrak{m}_{A'}) \rightarrow (A, \mathfrak{m}_A)$ is a surjective morphism in $\mathbf{Art}_{W(k)}(k)$ and S' is a flat A' -algebra. Then any element $f' \in S'$ is a non-zero divisor if and only if its reduction $f \in S = S' \otimes_{A'} A$ is. Moreover, if f' is a non-zero divisor then $S'/(f')$ is A' -flat.*

Proof. We begin by proving that if $f' \in S'$ is a non-zero divisor then its reduction $\bar{f} \in S'/\mathfrak{m}_{A'} S'$ is and that $S'/(f')$ is A' -flat. For this purpose, by we first take a minimal integer $n > 0$ such that $\mathfrak{m}_{A'}^{n+1} = 0$ and a non-zero element $a \in \mathfrak{m}_{A'}^n$ generating a submodule (a) of A' isomorphic to $A'/\mathfrak{m}_{A'}$. By the A' -flatness of S' we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S' \otimes_{A'} (a) \simeq S'/\mathfrak{m}_{A'} S' & \longrightarrow & S' & \longrightarrow & S'/(a) & \longrightarrow & 0 \\ & & \uparrow \bar{f} \cdot & & \uparrow f' \cdot & & \uparrow [f] \cdot & & \\ 0 & \longrightarrow & S' \otimes_{A'} (a) \simeq S'/\mathfrak{m}_{A'} S' & \longrightarrow & S' & \longrightarrow & S'/(a) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & K & \longrightarrow & 0 & & & & \end{array}$$

which implies that \bar{f} is a non-zero divisor. Using the resolution of $S'/(f')$ induced by multiplication by f' we see that $\mathrm{Tor}^1(S'/(f'), A'/\mathfrak{m}_{A'}) \simeq K = 0$ and therefore $S'/(f')$ is A' -flat by Lemma 2.1.3.

We are now ready for the proof of the proposition. Analogously as above we observe that any surjection $A' \rightarrow A$ in $\mathbf{Art}_{W(k)}(k)$ can be decomposed as a sequence of surjections satisfying additional condition: the kernel I of $A' \rightarrow A$ is annihilated by $\mathfrak{m}_{A'}$, that is, it is a finite dimensional $A'/\mathfrak{m}_{A'}$ vector space. By twisting the sequences $0 \rightarrow S' \rightarrow S' \rightarrow S'/(f') \rightarrow 0$ and $0 \rightarrow I \rightarrow A' \rightarrow A' / I = A \rightarrow 0$ together we obtain a diagram

$$\begin{array}{ccccccc}
& & S' / (\mathfrak{m}_{A'}, \bar{f}) S' \otimes_k I & \longrightarrow & S' / (f') & \longrightarrow & S / (f) \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & S' / \mathfrak{m}_{A'} S' \otimes_k I & \longrightarrow & S' & \longrightarrow & S \longrightarrow 0 \\
& & \bar{f} \cdot \text{id}_I \uparrow & & f' \cdot \uparrow & & f \cdot \uparrow \\
0 & \longrightarrow & S' / \mathfrak{m}_{A'} S' \otimes_k I & \longrightarrow & S' & \longrightarrow & S \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & K_{A' / \mathfrak{m}_{A'}} & \xrightarrow{\subset} & K_{A'} & \longrightarrow & K_A.
\end{array}$$

Now if f' is a non-zero divisor then $K_{A'} = 0$ which implies, by the first part of the proof, that S' / f' is A' -flat, and hence the upper row of the diagram is exact. By the snake lemma this gives a short exact sequence $0 \rightarrow K_{A' / \mathfrak{m}_{A'}} \rightarrow K_{A'} \rightarrow K_A \rightarrow 0$. This implies that $K_A = 0$ and therefore f is a non-zero divisor in S . Conversely, if f is a non-zero divisor then $K_A = 0$ then $K_{A' / \mathfrak{m}_{A'}} \simeq K_{A / \mathfrak{m}_A} = 0$ and therefore $K_{A'} = 0$. \square

Before proceeding, we introduce the following

Definition 2.1.6. Let M be a module over a noetherian ring S . We say that f_1, \dots, f_n is a *regular sequence* for M if $(f_1, \dots, f_n)M \neq M$ and f_{i+1} is not a zero-divisor for $M / (f_1, \dots, f_i)M$, for $0 \leq i \leq n-1$.

As a simple corollary we obtain:

Corollary 2.1.7. Let $(A', \mathfrak{m}_{A'}) \rightarrow (A, \mathfrak{m}_A)$ be a surjection in $\mathbf{Art}_{W(k)}(k)$ and let S' be an A' -flat algebra. Then the elements $f'_1, \dots, f'_k \in S$ for $k \geq 1$ form a regular sequence in S' if and only if the reductions f_1, \dots, f_k form a regular sequence in $S = S' \otimes_{A'} A$.

Proof. The proof follows from Lemma 2.1.5 by induction with respect to the parameter k . \square

Moreover, we have:

Lemma 2.1.8. Let $(A', \mathfrak{m}_{A'}) \rightarrow (A, \mathfrak{m}_A)$ be a surjection in $\mathbf{Art}_{W(k)}(k)$ and let $I = (f_1, \dots, f_m)$ be an ideal in $A[x_1, \dots, x_n]$. Every A' -flat lifting of $A[x_1, \dots, x_n] / I$ is given by $A'[x_1, \dots, x_n] / I'$, where I' is generated a sequence of lifts of $\{f_i\}$. Conversely, if (f_1, \dots, f_m) is a regular sequence then for every ideal I' generated by the sequence of lifts of $\{f_i\}$ the ring $A'[x_1, \dots, x_n] / I'$ is an A' -lifting.

Proof. We first observe that using similar arguments as in Lemma 2.1.5 we can prove that a homomorphism $M' \rightarrow N'$ with N' being A' -flat is surjective if and only if its reduction $M' \otimes_{A'} A \rightarrow N' \otimes_{A'} A$ is surjective. Now we take an A' -flat lifting B' of $A[x_1, \dots, x_n] / I$ and lift the surjection $\pi_A: A[x_1, \dots, x_n] \rightarrow A[x_1, \dots, x_n] / I$ to a mapping $\pi_{A'}: A'[x_1, \dots, x_n] \rightarrow B'$. By the above observation we see that $\pi_{A'}$ is surjective and

therefore B' is generated as an A' -algebra by n elements lifting $x_i \in A[x_1, \dots, x_n]/I$. To prove that the kernel $I' = \text{Ker}(A'[x_1, \dots, x_n] \rightarrow B')$ is generated by the liftings of the generators f_i , we apply the above observation again for the inclusion $(f'_1, \dots, f'_m) \rightarrow I'$ where $f'_i \in I'$, for $1 \leq i \leq m$, are liftings of $f_i \in I$ (I' is A' -flat, since B' is). This finishes the proof of the first part of the lemma. For the second part, we apply Corollary 2.1.7. \square

In the special case of $A = W_2(k)$ we obtain the following results.

Corollary 2.1.9. *A $W_2(k)$ -module M is flat if and only if the annihilator $\text{Ann}_M(p) = \{m \in M : pm = 0\}$ of p in M is equal to pM , i.e., the natural mapping $M/pM \rightarrow pM$ given by $[m] \mapsto pm$ is injective.*

Proof. We observe that $pW_2(k) \simeq W_2(k)/p$ and therefore $pW_2(k) \otimes_{W_2(k)} M$ is isomorphic to M/pM . Then we use Proposition 2.1.4. \square

As a corollary we obtain:

Corollary 2.1.10. *A ring $B/W_2(k)$ is a flat lifting of A/k if and only if $B/pB \simeq A$ and $pB = \text{Ann}_B(p)$. Moreover, for any flat lifting $B/W_2(k)$ of A the quotient B/J is a flat lifting of A/I if and only if J is a lifting of I and $(p) \cap J = pJ$.*

Proof. The first part follows directly from Corollary 2.1.9. To prove the second part, we apply the following sequence of equivalences:

$$\text{Ann}_{B/J} p = p \cdot B/J \Leftrightarrow (J : (p)) = (J + p) \Leftrightarrow p(J : (p)) = p(J + p) \Leftrightarrow (p) \cap (J) = pJ, \quad (2.3)$$

where the middle one follows from the inclusions $(J : (p)) \supset (p) \subset (J + p)$ and the injectivity of the mapping $B/p \rightarrow pB$. \square

2.2 Properties of morphisms of schemes

Definition 2.2.1 (Regular closed immersion). We say that a morphism $Z \rightarrow X$ is a *regular closed immersion* if it is a closed immersion such that the corresponding ideal sheaf is affine locally defined by a regular sequence.

Definition 2.2.2 (lci morphism). Let S be a locally noetherian scheme. A morphism $f : X \rightarrow S$ is a *local complete intersection* if affine locally there exists a factorization of the form:

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{A}_S^n \\ & \searrow f & \downarrow p \\ & & S, \end{array}$$

where $p : \mathbb{A}_S^n \rightarrow S$ is the projection and $i : X \rightarrow \mathbb{A}_S^n$ is a regular closed immersion.

Definition 2.2.3. We say that a morphism of schemes $f : X \rightarrow Y$ is of *finite Tor dimension* if \mathcal{O}_X is a $f^{-1}\mathcal{O}_Y$ -module of finite Tor dimension.

For any morphism $f : X \rightarrow Y$ of finite Tor dimension the structural sheaf \mathcal{O}_X is in fact quasi-isomorphic to a bounded complex of flat $f^{-1}\mathcal{O}_Y$ -modules (see [Sta17, Tag 08CI]).

Remark 2.2.4. In the case of morphisms of quasi-projective schemes over a field by [FM81, II.1.2] we know that the following morphisms are of finite Tor dimension:

- a) morphism with a smooth target,
- b) flat morphisms,
- c) regular closed immersions.

2.3 Basics of derived categories and spectral sequences

We now proceed to some basic facts concerning derived categories. In particular, we address the case of unbounded complex as they arise naturally in the context of deformations of singularities.

Definition 2.3.1. We say that a complex $\mathcal{E}^\bullet \in D(\mathcal{O}_X)$ is *perfect* if there exists an affine covering $X = \bigcup_i \text{Spec } A_i$ such that $\mathcal{E}^\bullet|_{\text{Spec}(A_i)}$ is quasi-isomorphic to a finite complex of projective A_i -modules.

2.3.1 Base change for higher direct images

We now present a few results concerning derived inverse images and base change for complexes. Our considerations concerning unboundedness are based on [Sta17, Tag 06XW] which gives a comprehensive exposition of the fundamental work [Spa88]. Due to the lack of an adequate reference, we give a detailed proof of the following lemma.

Lemma 2.3.2. *Suppose $f : X \rightarrow Y$ is a morphism of finite Tor dimension. Then, for any $\mathcal{F}^\bullet \in D(\mathcal{O}_X)$ there exists a convergent spectral sequence:*

$$E_2^{pq} = L^p f^* (\mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow L^{p+q} f^* \mathcal{F}^\bullet.$$

Proof. This is an application of the spectral sequence of a double complex. Indeed, by [Sta17, Tag 08DE] we see that the total derived pullback $Lf^* \mathcal{F}^\bullet$ is given by the totalisation of the double complex $f^{-1} \mathcal{F}^\bullet \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{E}^\bullet$ where \mathcal{E}^\bullet is a bounded $f^{-1} \mathcal{O}_Y$ -flat resolution of \mathcal{O}_X existing by the assumption on f . By the classical result there exist a spectral sequence:

$$E_2^{pq} = \mathcal{H}^p \mathcal{H}^q(f^{-1} \mathcal{F}^\bullet \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{E}^\bullet) \Rightarrow \mathcal{H}^{p+q} \text{Tot}(f^{-1} \mathcal{F}^\bullet \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{E}^\bullet),$$

whose convergence follows from the fact that $f^{-1} \mathcal{F}^\bullet \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{E}^\bullet$ is a double complex supported in the bounded horizontal strip $\{(i, j) : j \in \{k : \mathcal{E}^k \neq 0\}\}$. The sheaves $\mathcal{H}^{p+q} \text{Tot}(f^{-1} \mathcal{F}^\bullet \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{E}^\bullet)$ are easily identified with $L^{p+q} f^* \mathcal{F}^\bullet$. Moreover, by the exactness of the functor f^{-1} and flatness of \mathcal{E}^\bullet (in fact, we use [Sta17, Tag 08DE] again), we see that

$$\mathcal{H}^p \mathcal{H}^q(f^{-1} \mathcal{F}^\bullet \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{E}^\bullet) \simeq \mathcal{H}^p(f^{-1} \mathcal{H}^q(\mathcal{F}^\bullet) \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{E}^\bullet) \simeq L^p f^* \mathcal{H}^q(\mathcal{F}^\bullet).$$

This gives the desired result. □

Lemma 2.3.3. *Let $f : X \rightarrow S$ be a flat morphism of schemes, and \mathcal{E} a complex in $D_{\mathbf{QCoh}}(X)$. Then for every cartesian diagram:*

$$\begin{array}{ccc} X_T & \xrightarrow{i_f} & X \\ \downarrow f_T & & \downarrow f \\ T & \xrightarrow{i} & S, \end{array}$$

the natural base change $\phi : Li^ Rf_* \mathcal{E} \xrightarrow{\sim} Rf_{T*} Li_f^* \mathcal{E}$ is an isomorphism.*

Proof. For the proof we refer to [Sta17, Tag 08IB]. □

Lemma 2.3.4. *Suppose $f : X \rightarrow Y$ is a morphism of noetherian schemes. Then, for every $\mathcal{F}^\bullet \in D_{\mathbf{Coh}}^-(Y)$ there exists a natural isomorphism $Lf^* R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_Y) \simeq R\mathcal{H}om(Lf^* \mathcal{F}^\bullet, \mathcal{O}_X)$.*

Proof. By [Sta17, Tag 08I3] we obtain a mapping:

$$Lf^* R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_Y) \rightarrow R\mathcal{H}om(Lf^* \mathcal{F}^\bullet, \mathcal{O}_X).$$

In order to prove that it is an isomorphism we can work locally and therefore we may assume that \mathcal{F}^\bullet is resolved as a complex of locally free sheaves. In this case, the result follows from the definition of Lf^* and the fact that $R\mathcal{H}om(-, \mathcal{O}_X)$ might be computed by a locally free resolution of the first argument. □

2.4 Local cohomology

In this section we recall the notion of local cohomology and present a few results necessary in what follows. Our exposition is based on Grothendieck's lectures written up by Hartshorne (see [Har67]).

We begin with a general definition of local cohomology of a sheaf along a closed subset of a topological space. Throughout this section X is a topological space, $Z \subset X$ is a closed subset and $j : U \rightarrow X$ is the inclusion of the complement $X \setminus Z$.

Definition 2.4.1. Let \mathcal{F} be a sheaf of abelian groups on X . For any open subset $V \subset X$ we define the functor $\Gamma_Z(V, -) : \mathbf{Sh}(X) \rightarrow \mathbf{Ab}$ of *sections on V supported along Z* by the formula:

$$\Gamma_Z(V, \mathcal{F}) \stackrel{\text{def}}{=} \text{Ker}(\Gamma(V, \mathcal{F}) \rightarrow \Gamma(V \cap U, \mathcal{F})).$$

The association $V \mapsto \Gamma_Z(V, \mathcal{F})$ defines a sheaf of abelian groups on X , and therefore it is natural to introduce:

Definition 2.4.2. Let \mathcal{F} be a sheaf of abelian groups on X . We define the functor $\underline{\Gamma}_Z(\mathcal{F}) : \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(X)$ of *subsheaf supported on Z* by the formula:

$$\underline{\Gamma}_Z(\mathcal{F}) \stackrel{\text{def}}{=} \text{sheaf given by the association } V \mapsto \Gamma_Z(V, \mathcal{F}).$$

The above definition can also be expressed as follows. We observe that the restriction homomorphism $\Gamma(V, \mathcal{F}) \rightarrow \Gamma(V \cap U, \mathcal{F})$ can be identified with a mapping $\mathcal{F}(V) \rightarrow j_* \mathcal{F}|_U(V)$ induced by the natural unit homomorphism $\mathcal{F} \rightarrow j_* \mathcal{F}|_U$. Consequently, $\underline{\Gamma}_Z(\mathcal{F})$ is in fact a sheaf fitting into the sequence:

$$0 \longrightarrow \underline{\Gamma}_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_* \mathcal{F}|_U. \quad (2.4)$$

The basic cohomological properties of $\Gamma_Z(V, -)$ and $\underline{\Gamma}_Z$ are given in the following:

Proposition 2.4.3. *For any open subset $V \subset X$ the functor $\Gamma_Z(V, -): \mathbf{Sh}(X) \rightarrow \mathbf{Ab}$ is left exact. As a corollary, the functor $\underline{\Gamma}_Z: \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(X)$ is also left exact.*

Proof. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be an exact sequence of abelian sheaves on X . By definition of $\Gamma_Z(V, -)$ and the left-exactness of $\Gamma(V, -)$ we have the following diagram:

$$\begin{array}{ccccccc} & & \Gamma_Z(V, \mathcal{F}) & \xrightarrow{i} & \Gamma_Z(V, \mathcal{G}) & \xrightarrow{j} & \Gamma_Z(V, \mathcal{H}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & \mathcal{H}(U) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}(V \cap U) & \longrightarrow & \mathcal{G}(V \cap U) & \longrightarrow & \mathcal{H}(V \cap U), \end{array}$$

with two bottom sequences exact. To prove our claim we need to show that i is injective and that $\text{Ker}(j) = \text{Im}(i)$. This is exactly the part of the proof of the snake lemma not involving the connecting homomorphism. The second part follows from the application of the result just proven for all open $V \subset X$. \square

As a consequence of the above, we can consider total right derived functors $R\Gamma_Z(V, -): D(X) \rightarrow D(\mathbf{Ab})$ and $R\underline{\Gamma}_Z: D(X) \rightarrow D(\mathbf{Ab})$, and their associated cohomology groups.

Definition 2.4.4. Let \mathcal{F} be an abelian sheaf or more generally an element of $D(X)$. We define i -th local cohomology groups of \mathcal{F} along Z on V by the formula:

$$H_Z^i(V, \mathcal{F}) \stackrel{\text{def}}{=} R^i \Gamma_Z(V, \mathcal{F}).$$

Moreover, we define i -th local cohomology sheaves of \mathcal{F} along Z by the formula:

$$\mathcal{H}_Z^i(\mathcal{F}) \stackrel{\text{def}}{=} R^i \underline{\Gamma}_Z(\mathcal{F}).$$

As an application of Grothendieck's spectral sequence to the composition $\Gamma_Z(V, -) = \Gamma(V, -) \circ \underline{\Gamma}_Z$, we obtain:

Proposition 2.4.5. *Let $V \subset X$ be an open subset and let \mathcal{F} be an abelian sheaf on X . Then we have the equality of derived functors $R\Gamma_Z(V, -) = R\Gamma \circ R\underline{\Gamma}_Z$. Moreover, there exists a convergent spectral sequence given by:*

$$E_2^{ij} = H^i(V, \mathcal{H}_Z^j(\mathcal{F})) \implies H_Z^{i+j}(V, \mathcal{F}).$$

Proof. To apply Grothendieck's spectral sequence for composition of derived functors we need to prove:

Lemma 2.4.6. *Suppose \mathcal{F} is an injective sheaf on X . Then $\underline{\Gamma}_Z(\mathcal{F})$ is $R\Gamma$ -acyclic.*

Proof. It suffices to show that $\underline{\Gamma}_Z(\mathcal{F})$ is flabby whenever \mathcal{F} is flabby. For this purpose, we consider two open subsets $W \subset V$ and the associated commutative diagram:

$$\begin{array}{ccc} \Gamma_Z(V, \mathcal{F}) & \longrightarrow & \Gamma_Z(W, \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(W) \\ \downarrow & & \downarrow \\ \mathcal{F}(V \cap U) & \longrightarrow & \mathcal{F}(W \cap U), \end{array}$$

2.4.1 Long exact sequence of local cohomology for an open immersion

Here, we present a useful result relating local cohomology supported in Z with cohomology on the complement $U = X \setminus Z$. We precede the actual result with a simple lemma.

Lemma 2.4.7. *Suppose \mathcal{F} is a flabby sheaf on X . Then the natural mapping $\mathcal{F} \rightarrow j_*\mathcal{F}|_U$ is surjective.*

Proof. Let $V \subset X$ be an open subset. As we mentioned above, the mapping $\mathcal{F}(V) \rightarrow j_*\mathcal{F}|_U(V)$ can be identified with the restriction homomorphism $\mathcal{F}(V) \rightarrow \mathcal{F}(V \cap U)$, and is therefore surjective by flabbiness. \square

Theorem 2.4.8 (Long exact sequence of local cohomology). *Let $U \subset X$ be an open subset and let \mathcal{F}^\bullet be an element of $D(X)$. Then there exists a distinguished triangle in $D(X)$:*

$$R\underline{\Gamma}_Z(\mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet \longrightarrow Rj_*\mathcal{F}_{|U}^\bullet \longrightarrow R\underline{\Gamma}_Z(\mathcal{F}^\bullet)[1].$$

In particular, if \mathcal{F} is a sheaf on X , we have the following exact sequence:

$$0 \longrightarrow \mathcal{H}_Z^0(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*\mathcal{F}|_U \longrightarrow \mathcal{H}_Z^1(\mathcal{F}) \longrightarrow 0,$$

and isomorphisms $R^i j_* \mathcal{F}|_U \simeq \mathcal{H}_Z^{i+1}(\mathcal{F})$, for $i \geq 1$.

Proof. For the first part of the theorem, we observe that by Lemma 2.4.7 for an injective resolution $\mathcal{I}^\bullet \rightarrow \mathcal{F}^\bullet$ the sequence of complexes:

$$0 \longrightarrow \underline{\Gamma_Z}(\mathcal{I}^\bullet) \longrightarrow \mathcal{I}^\bullet \longrightarrow j_*\mathcal{I}_{|U}^\bullet \longrightarrow 0$$

is exact. The total derived functor $R\underline{\Gamma}_Z(\mathcal{F}^\bullet)$ is computed based on injective resolutions and therefore we are done with the first part of the lemma. The second part follows from the long exact sequence of cohomology sheaves. \square

As a corollary we obtain:

Corollary 2.4.9. *For any open $V \subset X$ there exists an exact sequence of cohomology groups:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_Z^0(V, \mathcal{F}^\bullet) & \longrightarrow & H^0(V, \mathcal{F}^\bullet) & \longrightarrow & H^0(V \cap U, \mathcal{F}_{|V \cap U}^\bullet) \\
& & & & \searrow \alpha^0 & & \\
& & H_Z^1(V, \mathcal{F}^\bullet) & \longrightarrow & H^1(V, \mathcal{F}^\bullet) & \longrightarrow & H^1(V \cap U, \mathcal{F}_{|V \cap U}^\bullet) \\
& & & & & & \vdots \\
& & H_Z^n(V, \mathcal{F}^\bullet) & \longrightarrow & H^n(V, \mathcal{F}^\bullet) & \longrightarrow & H^n(V \cap U, \mathcal{F}_{|V \cap U}^\bullet).
\end{array}$$

In particular, if V is an affine scheme we have an exact sequence of $\mathcal{O}_V(V)$ -modules:

$$0 \longrightarrow H_Z^0(V, \mathcal{F}) \longrightarrow H^0(V, \mathcal{F}) \longrightarrow H^0(V \cap U, \mathcal{F}) \longrightarrow H_Z^1(V, \mathcal{F}) \longrightarrow 0,$$

and a sequence of isomorphisms $H^i(V \cap U, \mathcal{F}) \simeq H_Z^{i+1}(V, \mathcal{F})$, for $i \geq 1$.

Proof. The proof follows from the long exact sequence of cohomology sheaves associated with:

$$R\Gamma_Z(V, \mathcal{F}^\bullet) \longrightarrow R\Gamma(V, \mathcal{F}^\bullet) \longrightarrow R\Gamma(V, \mathcal{F}_{|U}^\bullet) \longrightarrow R\Gamma_Z(V, \mathcal{F}^\bullet)[1],$$

which arises from the first part of the lemma by means of the functor $R\Gamma(V, -)$ (here, we use Proposition 2.4.5). \square

2.4.2 Local cohomology of modules over noetherian rings

It turns out that affine locally the sheaf-theoretic notions described above can be expressed in purely algebraic fashion. Let R be a noetherian ring, let \mathfrak{a} be an ideal, and let M be an R -module. We now describe the notions given above in the case of an affine scheme $X = \text{Spec}(R)$, a closed subset $Z = V(\mathfrak{a})$ and a quasi-coherent sheaf $\mathcal{F} = \widetilde{M}$. Our considerations naturally lead to a concept of local cohomology $H_{\mathfrak{a}}^i(M)$ supported in the ideal $\mathfrak{a} \triangleleft R$ which is an important tool in commutative algebra.

We begin by proving that taking local cohomology sheaves does not lead us out of the realm of quasi-coherent sheaves.

Lemma 2.4.10. *Let X be a locally noetherian scheme, Z a closed subscheme and let \mathcal{F} be a quasi-coherent sheaf. Then the local cohomology sheaves $\mathcal{H}_Z^i(\mathcal{F})$ are quasi-coherent.*

Proof. Firstly, we observe that by the assumptions on X the open immersion $U \rightarrow X$ is quasi-compact and quasi-separated, and therefore higher direct images $R^i j_* \mathcal{F}_U$ are quasi-coherent. Then, by Corollary 2.4.9, we see that $\mathcal{H}_Z^i(\mathcal{F})$ are either kernels or cokernels of maps of quasi-coherent sheaves and thus quasi-coherent. \square

Using Proposition 2.4.5 and the above lemma we see that $\mathcal{H}_{V(\mathfrak{a})}^i(\text{Spec}(R), \widetilde{M})$ is isomorphic to the sheaf associated with an R -module $H^0(\text{Spec}(R), \mathcal{H}_{V(\mathfrak{a})}^i(\text{Spec}(R), \widetilde{M})) \simeq H_{V(\mathfrak{a})}^i(\text{Spec}(R), \widetilde{M})$. The last module $H_{V(\mathfrak{a})}^i(\text{Spec}(R), \widetilde{M})$ can be expressed in algebraic terms. For this goal, we introduce the following definition.

Definition 2.4.11. Let M be a module over a noetherian ring R , and let \mathfrak{a} be an ideal in R . We define the functor of \mathfrak{a} -torsion sections of M by the formula $\Gamma_{\mathfrak{a}}(M) = \{m \in M : \exists_n \mathfrak{a}^{n+1}m = 0\}$.

Lemma 2.4.12. *The module $\Gamma_{\mathfrak{a}}(M)$ is naturally isomorphic with $H_{V(\mathfrak{a})}^0(\text{Spec}(R), \widetilde{M})$.*

Proof. Assume $\mathfrak{a} = (f_1, \dots, f_m)$. By definition $H_{\mathfrak{a}}^0(R, M)$ is the module of $\text{Spec}(R)$ sections of \widetilde{M} vanishing on $\text{Spec}(R) \setminus V(\mathfrak{a})$. Using quasi-coherence of \widetilde{M} and the covering $\text{Spec}(R) \setminus V(\mathfrak{a}) = \bigcup_i \text{Spec}(R_{f_i})$ we see that those sections are exactly the kernel of the natural mapping

$$M \rightarrow \bigoplus_i M_{f_i}.$$

An element $m \in M$ lies in the kernel if and only if there exists a sequence of integers α_i , for $i = 1, \dots, m$, such that $f_i^{\alpha_i} \cdot m = 0$. By a simple combinatorial argument this is equivalent to $m \in \Gamma_{\mathfrak{a}}(M)$. \square

By a straightforward argument, we see that the functor $\Gamma_{\mathfrak{a}}$ is left exact and therefore we may consider its right derived functors $R^i\Gamma_{\mathfrak{a}}$. We call them *local cohomology modules supported in \mathfrak{a}* and denote by $H_{\mathfrak{a}}^i(M)$. We prove that the modules $H_{\mathfrak{a}}^i(M)$ are isomorphic to $H_{V(\mathfrak{a})}^i(\mathrm{Spec}(R), \widetilde{M})$.

Theorem 2.4.13. *There exists a natural isomorphism $H_{\mathfrak{a}}^i(M) \simeq H_{V(\mathfrak{a})}^i(\mathrm{Spec}(R), \widetilde{M})$.*

Proof. We apply the standard techniques of coeffacable functors (cf. Hartshorne ktoerys-tam rozdział). We claim that both functors $M \mapsto H_{\mathfrak{a}}^i(M)$ and $M \mapsto H_{V(\mathfrak{a})}^i(\mathrm{Spec}(R), \widetilde{M})$ are universal δ -functors. For the first functor, it is clear since it is a derived functor of $\Gamma_{\mathfrak{a}}$. For the second, we show coeffacability. For this goal, we just observe that for any injective module I the associated sheaf \widetilde{I} is flabby and therefore higher local cohomology sheaves supported in $V(\mathfrak{a})$ vanish. Consequently, to conclude we can apply Lemma 2.4.12 and the standard properties of universal δ -functors. \square

It turns out that there are even more explicit, algebraic ways to compute $H_{\mathfrak{a}}^i(M)$. Let (f_1, \dots, f_m) be a generating set of an ideal \mathfrak{a} . We define the Čech complex of (f_1, \dots, f_m) to be the complex of R -modules

$$\check{C}(f_1, \dots, f_m): R \xrightarrow{d_0} \bigoplus_{1 \leq i \leq m} R_{f_i} \xrightarrow{d_1} \bigoplus_{1 \leq i < j \leq m} R_{f_i f_j} \xrightarrow{d_2} \dots \xrightarrow{d_{m-1}} R_{f_1 \dots f_m}$$

with differentials defined by the association

$$d_k((r_{i_1, \dots, i_k})_{1 \leq i_1 < \dots < i_k \leq m}) = \left(\sum_{1 \leq j \leq k+1} (-1)^j r_{i_1, \dots, \widehat{i_j}, \dots, i_{k+1}} \right).$$

Lemma 2.4.14. *There exist natural isomorphisms $\Gamma_{\mathfrak{a}}(M) = H^0(\check{C}(f_1, \dots, f_m) \otimes_R M)$ and $\Gamma_{\mathfrak{a}}(M) = \mathrm{colim}_{n \in \mathbb{N}} \mathrm{Hom}_R(R/\mathfrak{a}^{n+1}, M)$.*

Proof. It is straightforward. \square

Theorem 2.4.15. *Let M be a module over a noetherian ring R , and let $\mathfrak{a} = (f_1, \dots, f_m)$ an ideal in R . Then there exist natural isomorphisms $H_{\mathfrak{a}}^i(R, M) \simeq H^i(\check{C}(f_1, \dots, f_m) \otimes_R M)$ and $H_{\mathfrak{a}}^i(R, M) \simeq \mathrm{colim}_{n \in \mathbb{N}} \mathrm{Ext}_R^i(R/\mathfrak{a}^{n+1}, M)$.*

Proof. For the complete proof, see [Har67, Theorem 2.3]. Again, the basic idea is to use the standard techniques of universal δ -functor and the above Lemma 2.4.14. \square

Example 2.4.16. Let $R = k[x_1, \dots, x_n]$ be the polynomial ring and let $\mathfrak{m} = (x_1, \dots, x_n)$ be the ideal of the origin. Then $H^n(R) = x_1^{-1} \dots x_n^{-1} k[x_1^{-1}, \dots, x_n^{-1}]$, where x_i acts on $x_1^{-j_1} x_2^{-j_2} \dots x_n^{-j_n}$ by increasing j_i by one if $j_i < -1$ and by multiplying by 0 otherwise.

As a simple corollary we have the following proposition (see [ILL⁺07, Proposition 7.3]).

Proposition 2.4.17. *Let M be a module over a noetherian ring R , and let \mathfrak{a} be an ideal. The local cohomology modules $H_{\mathfrak{a}}^i(M)$ satisfy the following properties:*

- 1) *the module $H_{\mathfrak{a}}^0(R, M)$ is isomorphic to $\Gamma_{\mathfrak{a}}(M)$, and $H_{\mathfrak{a}}^j(R, M)$ is \mathfrak{a} -torsion for each $j \in \mathbb{N}$,*
- 2) *if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ then $H_{\mathfrak{a}}^i(R, M) = H_{\mathfrak{b}}^i(R, M)$,*

3) an exact sequence of R -modules $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ induces a long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{a}}^i(R, M) \longrightarrow H_{\mathfrak{a}}^i(R, M') \longrightarrow H_{\mathfrak{a}}^i(R, M'') \longrightarrow H_{\mathfrak{a}}^{i+1}(R, M) \longrightarrow \cdots.$$

Proof. For the proof, [ILL⁺07, Proposition 7.3]). □

2.4.3 Depth and local cohomology, S_n properties

One of the most important properties of local cohomology is expressed by the following theorem which it with the following notion of depth, and consequently with important concepts of Gorenstein and Cohen-Macaulay rings.

Definition 2.4.18. Let M be a module over a noetherian ring R , and let \mathfrak{a} be an ideal such that $\mathfrak{a}M \neq M$. We define the *depth of M with respect to \mathfrak{a}* as the length of a longest regular sequence contained in \mathfrak{a} . We denote it by $\text{depth}_{\mathfrak{a}}(M)$.

It turns out that $\text{depth}_{\mathfrak{a}}(M)$ is determined by the local cohomology $H_{\mathfrak{a}}^i(R, M)$. More precisely we have the following theorem:

Theorem 2.4.19 ([ILL⁺07, Theorem 9.1]). *Let M be a finitely generated module over a noetherian ring R , and let \mathfrak{a} be an ideal such that $\mathfrak{a}M \neq M$. Then $\text{depth}_{\mathfrak{a}}(M) = \inf\{n: H_{\mathfrak{a}}^n(R, M) \neq 0\}$.*

Definition 2.4.20. Let R be a noetherian ring and let M be a finitely generated R -module. We say that M satisfies property S_n if for any prime ideal $\mathfrak{p} \in \text{Spec}(R)$ we have:

$$\text{depth}_{\mathfrak{p}} M \geq \min(n, \dim M_{\mathfrak{p}}).$$

Moreover, we say that a coherent sheaf on a scheme satisfies property S_r if every of its stalks does.

Theorem 2.4.21 (Serre's normality criterion). *A noetherian ring R is normal if and only if R is regular in codimension 1 and satisfies property S_2 .*

Definition 2.4.22. We say that a local ring (R, \mathfrak{m}) is *Cohen-Macaulay* if $\text{depth}_{\mathfrak{m}}(R) = \dim(R)$. Moreover, we say that a scheme X is *Cohen-Macaulay* if for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.

The property of being Cohen-Macaulay can be expressed in terms of vanishing of local cohomology. Namely, we have

Proposition 2.4.23. *A local ring (R, \mathfrak{m}) is Cohen-Macaulay if and only $H_{\mathfrak{m}}^i(R) = 0$ for $i < \dim R$.*

Proof. We use Theorem 2.4.19. □

In our considerations concerning deformations of open subsets and their Frobenius morphism, we need the following:

Lemma 2.4.24 ([HK04, Proposition 3.3]). *Let X be a scheme and satisfying property S_r . Let $Z \subset X$ be a closed subscheme such that $\text{codim}(Z, X) \geq r$. Then we have $\mathcal{H}_Z^i(X, \mathcal{O}_X) = 0$, for $i < r$.*

2.5 Cones over projective schemes

Let X be a proper scheme and let \mathcal{L} be an invertible sheaf on X . The *total space* $\text{Tot}_{X,\mathcal{L}}$ of the line bundle associated to \mathcal{L} is defined in terms of functor of points by the formula:

$$\mathbf{Sch}_k \ni S \mapsto \{f : S \rightarrow X, s \in H^0(S, f^*\mathcal{L})\}.$$

The above functor is representable by a scheme over X given by $\text{Spec}_X \oplus_{n \geq 0} \mathcal{L}^{\otimes -n}$ equipped with a natural affine projection $p_{X,\mathcal{L}} : \text{Tot}_{X,\mathcal{L}} \rightarrow X$. The zero section $Z_{X,\mathcal{L}} \subset \text{Tot}_{X,\mathcal{L}}$ is a subscheme associated with the surjection $\oplus_{n \geq 0} \mathcal{L}^{\otimes -n} \rightarrow \mathcal{O}_X$ and consequently the scheme $\text{Tot}_{X,\mathcal{L}}^* \stackrel{\text{def}}{=} \text{Tot}_{X,\mathcal{L}} \setminus Z_{X,\mathcal{L}}$ is isomorphic to $\text{Spec}_X \oplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}$. The *section ring* $R_{X,\mathcal{L}}$ of \mathcal{L} is defined by the formula:

$$R_{X,\mathcal{L}} \stackrel{\text{def}}{=} \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n}).$$

The ring $R_{X,\mathcal{L}}$ is naturally graded by the parameter n , and the irrelevant ideal $\mathfrak{m}_{X,\mathcal{L}}$ is given by all sections of positive powers of \mathcal{L} . The ring of global section of the sheaf $\mathcal{O}_{\text{Tot}_{X,\mathcal{L}}^*}$ is naturally isomorphic with $R_{X,\mathcal{L}}$ and therefore we obtain a mapping $\text{Tot}_{X,\mathcal{L}}^* \rightarrow \text{Spec}(R_{X,\mathcal{L}})$.

Theorem 2.5.1 (Grauert's ampleness criterion). *Let X be a proper scheme over k and let \mathcal{L} be a line bundle on X . Then \mathcal{L} is ample if and only if the natural mapping $\text{Tot}_{X,\mathcal{L}}^* \rightarrow \text{Spec}(R_{X,\mathcal{L}})$ is an open immersion onto $\text{Spec}(R_{X,\mathcal{L}}) \setminus \{\mathfrak{m}_{X,\mathcal{L}}\}$.*

Proof. For the proof, see [Gro61, 8.9.1]. □

Motivated by the above theorem, we define the *cone over X associated with \mathcal{L}* to be the scheme $\text{Cone}_{X,\mathcal{L}} \stackrel{\text{def}}{=} \text{Spec}(R_{X,\mathcal{L}})$. The basic properties of $\text{Cone}_{X,\mathcal{L}}$ are summarized by the following

Proposition 2.5.2. *Let X be a proper scheme and let \mathcal{L} be an ample line bundle. Then the local cohomology $H_{\mathfrak{m}_{X,\mathcal{L}}}^i(R_{X,\mathcal{L}})$ groups satisfy the following properties*

- 1) $H_{\mathfrak{m}_{X,\mathcal{L}}}^0(R_{X,\mathcal{L}}) = H_{\mathfrak{m}_{X,\mathcal{L}}}^1(R_{X,\mathcal{L}}) = 0$,
- 2) $H_{\mathfrak{m}_{X,\mathcal{L}}}^{i+1}(R_{X,\mathcal{L}}) = \bigoplus_{k \in \mathbb{Z}} H^i(X, \mathcal{O}_X(k))$, for $i \geq 1$.

In particular, the scheme $\text{Cone}_{X,\mathcal{L}}$ is normal if and only if X is, and it is Cohen-Macaulay if and only if $H^i(X, \mathcal{O}_X(k)) = 0$, for any $i \geq 1$ and $k \in \mathbb{Z}$.

Proof. The first part of the proof we apply long exact sequence of local cohomology (cf. Cor. 2.4.9) to the inclusion $\text{Spec}(R_{X,\mathcal{L}}) \setminus \{\mathfrak{m}\} \simeq \text{Tot}_{X,\mathcal{L}}^*$. The rest follows from Theorem 2.4.21 and Proposition 2.4.23. □

Remark 2.5.3. In fact, for $i \geq 1$ the local cohomology groups $H_{\mathfrak{m}_{X,\mathcal{L}}}^{i+1}(R_{X,\mathcal{L}})$ are naturally \mathbb{Z} -graded with

$$[H_{\mathfrak{m}_{X,\mathcal{L}}}^{i+1}(R_{X,\mathcal{L}})]_k = H^i(X, \mathcal{O}_X(k)), \text{ for } k \in \mathbb{Z}$$

2.5.1 Relation to the cone over a given projective embedding

Here, we relate the above considerations with the natural construction of a cone over a projective embedding $X \subset \mathbb{P}^n$. Let \mathcal{I}_X be the ideal sheaf of X inside \mathbb{P}^n . We define the cone over the projective embedding of $X \subset \mathbb{P}^n$, denoted in what follows by $\text{Cone}_{X, \mathbb{P}^n}$, as the quotient $\Gamma^+(\mathcal{O}_{\mathbb{P}^n})/\Gamma^+(\mathcal{I}_X)$, where $\Gamma^+(\mathcal{F}) \stackrel{\text{def}}{=} \bigoplus_{k \in \mathbb{N}} H^0(\mathbb{P}^n, \mathcal{F}(k))$.

Proposition 2.5.4. *The cone over the embedding $\text{Cone}_{X, \mathbb{P}^n}$ is isomorphic to $\text{Cone}_{X, \mathcal{O}_X(1)}$ if and only if the embedding $X \subset \mathbb{P}^n$ is projectively normal.*

Proof. By the long exact sequence of cohomology we see that

$$0 \longrightarrow \bigoplus_{k \in \mathbb{N}} H^0(\mathbb{P}^n, \mathcal{I}_X(k)) \longrightarrow \bigoplus_{k \in \mathbb{N}} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \longrightarrow \bigoplus_{k \in \mathbb{N}} H^0(X, \mathcal{O}_X(k)) \longrightarrow \bigoplus_{k \in \mathbb{N}} H^1(\mathbb{P}^n, \mathcal{I}_X(k)) \longrightarrow \dots,$$

is exact and therefore there is a natural inclusion $\Gamma^+(\mathcal{O}_{\mathbb{P}^n})/\Gamma^+(\mathcal{I}_X) \rightarrow R_{X, \mathcal{O}_X(1)}$, inducing the normalization of $\text{Cone}_{X, \mathbb{P}^n}$, which is an isomorphism if and only if the natural restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \longrightarrow \bigoplus_{k \in \mathbb{N}} H^0(X, \mathcal{O}_X(k))$ is surjective, that is, the embedding $X \subset \mathbb{P}^n$ is projectively normal. \square

Chapter 3

Deformation theory

In the following chapter, we give a short overview of deformation theory problems together with a handful of tools necessary to tackle them. We present general abstract theory and defer most of the examples and applications to subsequent chapters.

First, in §3.1 we recall the construction and basic properties of the cotangent complex. Then, in §3.2, we present the most important questions of deformation theory, such as deformation of schemes and deformation of morphisms. Moreover, we explain how cotangent complex can be applied to address these questions. In §3.4, we give a few technical results concerning functoriality of obstruction classes, and in §3.5 we summarise the theory in the case of affine schemes. The penultimate §3.6 contains a thorough description of the formalism of functors of Artin rings, which gives a natural framework for tackling problems in deformation theory.

Finally, in §3.7, we present the first abstract results concerning deformations of the Frobenius morphism.

3.1 The cotangent complex

Let $f: X \rightarrow Y$ be a morphism of schemes over a base scheme S . The sheaves of relative Kähler differentials give rise to an exact sequence of \mathcal{O}_X -modules

$$f^*\Omega_{Y/S} \longrightarrow \Omega_{X/S}^1 \longrightarrow \Omega_{X/Y}^1 \longrightarrow 0 \quad (3.1)$$

which is left-exact if f is smooth. The *cotangent complex* $L_{X/Y} \in D_{\mathbf{QCoh}}^-(\mathcal{O}_X)$ is a derived generalisation of the sheaf of Kähler differentials that allows one to circumvent problems with left non-exactness of the above sequence. More precisely, every morphism $f: X \rightarrow Y$ leads to a distinguished triangle

$$Lf^*L_{Y/S} \longrightarrow L_{X/S} \longrightarrow L_{X/Y} \longrightarrow Lf^*L_{Y/S}[1],$$

which in turn induces a long exact sequence of cohomology extending (3.1) to the left, since $\mathcal{H}^0(L_{X/Y}) \simeq \Omega_{X/Y}^1$. The construction is based on techniques from simplicial algebra introduced independently by Quillen and André (leading to so-called André-Quillen cohomology of rings) and extended to the setting of algebraic geometry by Illusie [Ill71]. In fact, it works for every morphism $\mathcal{A} \rightarrow \mathcal{B}$ of rings in a topos yielding the cotangent complex $L_{\mathcal{B}/\mathcal{A}}$. The cotangent complex $L_{X/Y}$ is obtained as the particular case $L_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$. We describe the construction in a purely algebraic setting, that is, in the case of sheaves on a one-point site $\{*\}$. By the standard anti-equivalence between categories of affine schemes and rings, this is sufficient for the considerations involving

affine schemes. The reader is referred to [Sta17, Chapter 77] for a modern treatment of the subject.

3.1.1 Construction for affine schemes

Let $A \rightarrow B$ be a ring map. Let $U: \mathbf{Alg}_A \rightarrow \mathbf{Sets}$ be the forgetful functor from the category of A -algebras to the category of sets, and let $F: \mathbf{Sets} \rightarrow \mathbf{Alg}_A$ be the functor of a free algebra, left-adjoint to U , defined by the formula

$$\mathbf{Sets} \ni E \mapsto A[E] \in \mathbf{Alg}_A.$$

Since U and F are adjoint, there exist a co-unit natural transformation $\mathrm{ev}_{B/A}: FU \rightarrow \mathrm{id}_{\mathbf{Alg}_A}$, defined for an object $B \in \mathbf{Alg}_A$ as the evaluation $A[B] \ni b \mapsto b \in B$, and a unit natural transformation $\iota_{E/A}: \mathrm{id}_{\mathbf{Sets}} \rightarrow UF$, defined for an object $E \in \mathbf{Sets}$ as the inclusion $E \ni e \mapsto e \in A[E]$. For every A -algebra B , the data just described gives rise to a simplicial A -algebra P_\bullet . On the level of objects P_\bullet is defined by the formula

$$P_n = (FU)^{n+1}(B) = \underbrace{A[A[\dots A[B] \dots]]}_{(n+1) \text{ times}}.$$

The face maps $\delta_i: P_n \rightarrow P_{n-1}$ are given by the formulas $\delta_i = (FU)^{n-i} \circ \mathrm{ev}_{(FU)^i(B)}$, i.e., by the evaluation on the i -th level of the compound $(FU)^{n+1}(B)$. Similarly, degeneracy maps $\sigma_j: P_{n-1} \rightarrow P_n$ are defined as $\sigma_j = (FU)^{n-j} \circ \iota_{(FU)^j(B)}$, that is, as the inclusion on the i -th level of the compound $(FU)^n(B)$.

Each of A -algebras P_n admits a natural augmentation map $\varepsilon: P_n \rightarrow B$ induced by the $(n+1)$ -fold evaluation. The cotangent complex $L_{B/A}$ is defined as the cochain complex (see the Dold-Kan correspondence in [Wei94]) associated with the simplicial B -module $\Omega_{P_\bullet/A}^1 \otimes_\varepsilon B$. Similar construction, in fact with U and F substituted with their sheafified versions, works in general for every morphism $\mathcal{A} \rightarrow \mathcal{B}$ of rings in a topos.

Theorem 3.1.1 ([Sta17, Tag 08T3]). *The cotangent complex of a morphism of affine schemes $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ is quasi-isomorphic to the sheafification $\widetilde{L_{B/A}}$.*

3.1.2 Properties of the cotangent complex

We now focus on the cotangent complex of morphisms of schemes. The main properties of $L_{X/Y}$ are given by the following theorem which summarizes the results of [Ill71].

Theorem 3.1.2. *The cotangent complex of a morphism of schemes satisfies the following properties.*

- (1) *There exists a natural morphism $L_{X/Y} \rightarrow \Omega_{X/Y}^1$ in $D(\mathcal{O}_X)$ inducing an isomorphism $\tau_{\geq 0} L_{X/Y} \simeq \Omega_{X/Y}^1$, the morphism is an isomorphism if $X \rightarrow Y$ is smooth.*
- (2) *If $X \rightarrow Y$ is a lci morphism (cf. Definition 2.2.2), then the complex $L_{X/Y}$ is a perfect complex (cf. Definition 2.3.1) supported in degrees $[-1, 0]$.*
- (3) *There exists a natural morphism $df: Lf^* L_{Y/S} \rightarrow L_{X/S}$ (called the differential of f) which fits into a distinguished triangle*

$$Lf^* L_{Y/S} \longrightarrow L_{X/S} \longrightarrow L_{X/Y} \longrightarrow Lf^* L_{Y/S}[1].$$

Moreover, for every pair of morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ the differentials satisfy the equality $d(g \circ f) = df \circ Lf^ dg$.*

(4) If $X \rightarrow Y$ is a morphism locally of finite type with Y locally noetherian then $L_{X/Y} \in D_{\text{Coh}}^-(\mathcal{O}_X)$.

Proof. The proofs of consecutive assertions are given in [Ill71, Chapitre II, Corollaire 1.2.4.2], [Ill71, Chapitre II, something], [Ill71, Chapitre II, Proposition 2.1.2] and [Ill71, Chapitre II, Corollaire 2.3.7], respectively. \square

Remark 3.1.3. In fact, in (2) the complex $L_{X/Y}$ is locally quasi-isomorphic to $I/I^2 \rightarrow \Omega_{P/A}^1 \otimes_P B$, where P is the polynomial algebra over A in a finite number of variables and $P \rightarrow B$ is a surjective morphism with kernel I generated by a regular sequence (cf. Definition 2.2.2).

The morphism $L_{X/Y} \rightarrow Lf^*L_{Y/S}[1]$ is called the *Kodaira–Spencer class* of f . It satisfies the following property.

Proposition 3.1.4 (Composition of Kodaira–Spencer). *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be a pair of morphisms over S . Then the Kodaira–Spencer classes $K_{X/Z/S}$ and $K_{Y/Z/S}$ satisfy the relation $Lf^*K_{Y/Z/S} = K_{X/Z/S} \circ df$, where $u = g \circ f$*

Proof. The proof follows from the commutativity of diagram

$$\begin{array}{ccccccc}
 Lu^*L_{Z/S} \simeq Lf^*Lg^*L_{Z/S} & \xrightarrow{Lf^*dg} & Lf^*L_{Y/S} & \longrightarrow & Lf^*L_{Y/Z} & \xrightarrow{Lf^*K_{Y/Z/S}} & Lu^*L_{Z/S}[1] \\
 \parallel & & \downarrow df & & \downarrow df & & \parallel \\
 Lu^*L_{Z/S} & \xrightarrow{du} & L_{X/S} & \longrightarrow & L_{X/Z} & \xrightarrow{K_{X/Z/S}} & Lu^*L_{Z/S}[1]
 \end{array}$$

induced by the naturality of the differential morphism (see Theorem 3.1.2 (3)). \square

Similar to sheaves of Kähler differentials, the cotangent complex is well-behaved with respect to base change.

Theorem 3.1.5 ([Ill71, Chapitre II, Proposition 2.2.3]). *Let $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ be morphisms over a base S . Let $p: X \times_Z Y \rightarrow X$ and $q: X \times_Z Y \rightarrow Y$ denote the projections. Suppose that either f or g is flat.*

- (1) *The natural map $Lp^*L_{X/Z} \rightarrow L_{X \times_Z Y/Z}$ given by the composition of dp and $L_{X \times_Z Y/Z} \rightarrow L_{X \times_S Y/Y}$ is an isomorphism.*
- (2) *The direct sum of maps dp and dq yields an isomorphism $Lp^*L_{Y/Z} \oplus Lq^*L_{X/Z} \simeq L_{X \times_Z Y/Z}$.*

The Kodaira–Spencer class also satisfies the following additivity property.

Proposition 3.1.6 (Additivity of Kodaira–Spencer). *Let $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ be morphisms of S -schemes. Let $p: X \times_Z Y \rightarrow X$ and $q: X \times_Z Y \rightarrow Y$ denote the projections, $u: X \times_Z Y \rightarrow Z$ the composition $u = f \circ p = g \circ q$. Then Kodaira–Spencer class*

$$K_{X \times_Z Y/Z/S} \in \text{Ext}^1(L_{X \times_Z Y/Z}, Lu^*L_{Z/S})$$

equals the direct sum of pullbacks of Kodaira–Spencer classes:

$$K_{X/Z/S} \in \text{Ext}^1(L_{X/Z}, Lf^*L_{Z/S}) \quad \text{and} \quad K_{Y/Z/S} \in \text{Ext}^1(L_{Y/Z}, Lg^*L_{Z/S}).$$

Proof. Using Proposition 3.1.4, we see that $K_{X \times_Z Y/Z/S} \circ dp = Lp^*K_{X/Z/S}$ and $K_{X \times_Z Y/Z/S} \circ dq = Lq^*K_{Y/Z/S}$. Adding these equations up, we obtain $K_{X \times_Z Y/Z/S} \circ (dp + dq) = Lp^*K_{X/Z/S} + Lq^*K_{Y/Z/S}$. By Theorem 3.1.5 (2), the morphism $dp + dq$ yields an isomorphism $Lp^*L_{Y/Z} \oplus Lq^*L_{X/Z} \simeq L_{X \times_Z Y/Z}$ and therefore the assertion is proved. \square

3.2 Classical problems of deformation theory

We adopt the following convention. Let \mathcal{B} be a sheaf of rings on a site, and let \mathcal{I} be an ideal sheaf. We say that \mathcal{I} is of *square zero* if $\mathcal{I}^2 = 0$ inside \mathcal{B} . Moreover, we say that a closed immersion $X \rightarrow \tilde{X}$ is a *first-order thickening* or is of *square zero* if the ideal \mathcal{I} of X inside \tilde{X} is of square-zero. In this case, \mathcal{I} admits a natural structure of an \mathcal{O}_X -module. Throughout this chapter, \mathfrak{S} denotes a site (see [Mil80, Section 5]).

3.2.1 Extension of sheaves of algebras

Let $f: X \rightarrow S$ be a morphism of schemes, and let \mathcal{I} be an \mathcal{O}_X -module. We define a technical notion of an S -extension of X by \mathcal{I} .

Definition 3.2.1 (Extensions of sheaves algebras). An S -extension of X by \mathcal{I} is a closed immersion $X \rightarrow X'$ of square-zero of schemes over a base S together with an identification $\mathcal{I} \simeq \text{Ker}(\mathcal{O}_{X'} \rightarrow \mathcal{O}_X)$.

Extensions of X by \mathcal{I} admit a natural structure of a category, denoted by $\underline{\text{Exal}}_S(X, \mathcal{I})$, with morphisms given by diagrams morphisms $X'_1 \rightarrow X'_2$ compatible with the closed immersions $X \rightarrow X'_1$ and $X \rightarrow X'_2$. We denote the set of isomorphism classes in $\underline{\text{Exal}}_S(X, \mathcal{I})$ by $\text{Exal}_S(X, \mathcal{I})$.

Remark 3.2.2. It turns out that extensions can be defined purely algebraically. More precisely, there exists a one-to-one correspondence between S -extensions of X by \mathcal{I} and diagrams of sheaves on X of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & & & \uparrow & \nearrow & \\ & & & & f^{-1}\mathcal{O}_S & & \end{array}$$

From this perspective, a morphism of extensions $\varphi: X'_1 \rightarrow X'_2$ corresponds to a homomorphism of $f^{-1}\mathcal{O}_X$ -algebras filling the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{X'_1} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi^\# & & \parallel \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{X'_2} & \longrightarrow & \mathcal{O}_X \longrightarrow 0, \end{array}$$

and therefore, by the five lemma, is an isomorphism. Consequently, $\underline{\text{Exal}}_S(X, \mathcal{I})$ is a groupoid.

The categories of extensions satisfy certain functoriality properties. First, for every homomorphism of \mathcal{O}_X -modules $w: \mathcal{I} \rightarrow \mathcal{I}'$ there exists a functor

$$w_\times: \underline{\text{Exal}}_S(X, \mathcal{I}) \rightarrow \underline{\text{Exal}}_S(X, \mathcal{I}'),$$

which associates to a given extension $\mathcal{O}_{X'} \in \underline{\text{Exal}}_S(X, \mathcal{I})$, an extension defined by the push-out diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \downarrow w & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & w_\times \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0, \end{array}$$

that is, by the formula $w_\times \mathcal{O}_{X'} \stackrel{\text{def}}{=} \mathcal{O}_{X'} \oplus \mathcal{I}' / \mathcal{I}$, where $\mathcal{I} \subset \mathcal{O}_{X'} \oplus \mathcal{I}'$ is the ideal given by the anti-diagonal embedding:

$$\mathcal{I} \rightarrow \mathcal{O}_{X'} \oplus \mathcal{I}', \quad x \mapsto (x, -w(x)).$$

Second, for every scheme morphism $f: X \rightarrow Y$ over S there exists a functor

$$f_\times: \underline{\text{Exal}}_S(X, \mathcal{I}) \rightarrow \underline{\text{Exal}}_S(Y, f_* \mathcal{I})$$

which associates to a given extension $\mathcal{O}_{X'} \in \underline{\text{Exal}}_S(X, \mathcal{I})$, an extension defined by the pull-back product diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & f_* \mathcal{I} & \longrightarrow & f_\times \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_Y \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow f^\# \\ 0 & \longrightarrow & f_* \mathcal{I} & \longrightarrow & f_* \mathcal{O}_{X'} & \longrightarrow & f_* \mathcal{O}_X \longrightarrow 0. \end{array}$$

Remark 3.2.3. For any ring homomorphism $A \rightarrow B$ and a B -module I , the above algebraic description leads to a natural notion of the category of A -extensions of B by I . We denote this category by $\underline{\text{Exal}}_A(B, M)$, and the set of isomorphisms classes by $\text{Exal}_A(B, M)$. Clearly, there is an equivalence between $\underline{\text{Exal}}_A(B, M)$ and $\underline{\text{Exal}}_{\text{Spec}(A)}(\text{Spec}(B), \tilde{I})$.

Example 3.2.4. Note that any first-order thickening $X \rightarrow \tilde{X}$ of kernel \mathcal{I} gives rise to an extension $\mathcal{O}_{\tilde{X}} \in \text{Exal}_{\mathbb{Z}}(\mathcal{O}_X, \mathcal{I})$. Algebraically, for any perfect field k of characteristic $p > 0$, the ring surjections $W_2(k) \rightarrow k$ and $k[\varepsilon]/(\varepsilon^2) \rightarrow k$ give rise to non-equal elements in $\text{Exal}_{\mathbb{Z}}(k, k)$.

More generally, we have the following

Definition 3.2.5. Let $f: X \rightarrow Y$ be a morphism of S schemes, and let $X \rightarrow X'$ (resp. $Y \rightarrow Y'$) be an S -extension of X (resp. Y) by the ideal \mathcal{I} (resp. \mathcal{J}). Moreover, let $w: f^* \mathcal{J} \rightarrow \mathcal{I}$ be a morphism of \mathcal{O}_X -modules. We define a w -compatible morphism of extensions X' and Y' as a commutative diagram of schemes

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow f & & \downarrow f' \\ Y & \longrightarrow & Y'. \end{array}$$

such that the natural map $f^{-1} \mathcal{J} \rightarrow \mathcal{I}$ in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \uparrow & & \uparrow f'^\# & & \uparrow f^\# \\ 0 & \longrightarrow & f^{-1} \mathcal{J} & \longrightarrow & f^{-1} \mathcal{O}_{Y'} & \longrightarrow & f^{-1} \mathcal{O}_Y \longrightarrow 0 \end{array}$$

is equal to a mapping $w^\circ: f^{-1} \mathcal{J} \rightarrow \mathcal{I}$ associated with $w: f^* \mathcal{J} \rightarrow \mathcal{I}$.

Remark 3.2.6. Suppose \mathcal{O}_X is a flat \mathcal{O}_Y -module. We say that a morphism $X' \rightarrow Y'$ is a flat extension of X over Y' (cf. Definition 1.4.1) if the natural map $f^* \mathcal{J} \rightarrow \mathcal{I}$ is an isomorphism.

Remark 3.2.7. As in Remark 3.2.3, the algebraic considerations above naturally lead to a notion of a compatible morphism of ring extensions.

It turns out that extensions can be classified using cotangent complex. More precisely, we have the following.

Theorem 3.2.8 ([Ill71, Chapitre III, Théorème 1.2.3]). *For any scheme X over a base S and an \mathcal{O}_X -module \mathcal{I} there exists a bijection between the group $\text{Exal}_S(X, \mathcal{I})$ of isomorphism classes of S -extension of X by an \mathcal{O}_Y -module \mathcal{I} and the group $\text{Ext}^1(L_{X/S}, \mathcal{I})$.*

Remark 3.2.9. For any morphism $w: \mathcal{I} \rightarrow \mathcal{I}'$, the bijection is compatible with the natural functorial structures of $\text{Exal}_S(X, -)$ and $\text{Ext}^1(L_{X/S}, -)$. Moreover, for any morphism of schemes $f: X \rightarrow Y$ the following compatibility holds:

$$\begin{array}{ccc} \text{Exal}_S(X, \mathcal{I}) & \xrightarrow{\quad\quad\quad} & \text{Ext}^1(L_{X/S}, \mathcal{I}) \\ \downarrow f_{\times} & & \downarrow \circ df \\ \text{Exal}_S(Y, f_*\mathcal{I}) & \longrightarrow \text{Ext}^1(L_{Y/S}, f_*\mathcal{I}) \longrightarrow \text{Ext}^1(L_{Y/S}, Rf_*\mathcal{I}) \simeq \text{Ext}^1(Lf^*L_{Y/S}, \mathcal{I}) & \end{array}$$

Remark 3.2.10. In the algebraic setting, for any ring homomorphism $A \rightarrow B$ and a B -module I there is a bijection between $\text{Exal}_A(B, I)$ and $\text{Ext}_B^1(L_{B/A}, I)$. The compatibility conditions are naturally translated.

3.2.2 Lifting algebras and lifting schemes

We proceed to the description of infinitesimal deformations of schemes. We begin with a technical definition.

Definition 3.2.11 (Deformation tuple). A *deformation tuple over S* consists of:

a) a diagram of schemes

$$\begin{array}{ccc} X & & \\ \downarrow f & & \\ Y & \xrightarrow{i} & Y' \\ \downarrow & \swarrow & \\ S, & & \end{array}$$

where Y' is a square-zero S -extension of Y by a module \mathcal{J} ,

b) an \mathcal{O}_X -module \mathcal{I} and a module homomorphism $w: f^*\mathcal{J} \rightarrow \mathcal{I}$.

Every deformation tuple as above gives rise to a natural notion of deformation of X over $Y \rightarrow Y'$. More precisely, we have the following

Definition 3.2.12. A *w -compatible deformation of X over $Y \rightarrow Y'$* is an S -extension of X by \mathcal{I} together with a w -compatible morphism of S -extensions $f': X' \rightarrow Y'$.

A w -compatible deformation can be illustrated by the following diagram

$$\begin{array}{ccc} X & \dashrightarrow & \tilde{X} \\ \downarrow f & & \downarrow \\ Y & \xrightarrow{i} & \tilde{Y} \\ \downarrow & \swarrow & \\ S, & & \end{array}$$

On the level of structural sheaves, a w -compatible deformation consists of a diagram of sheaves on X

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \uparrow w^\circ & & \uparrow f'^\# & & \uparrow f^\# \\ 0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{Y'} & \longrightarrow & f^{-1}\mathcal{O}_Y \longrightarrow 0. \end{array}$$

Remark 3.2.13. In the special case when $f: X \rightarrow Y$ is flat, $\mathcal{I} = f^*\mathcal{J}$ and $w: f^*\mathcal{J} \rightarrow \mathcal{I}$ is the natural isomorphism, the notion of a w -compatible deformation is nothing else but a flat deformation of X over Y' (cf. Definition 1.4.1). Indeed, using Proposition 2.1.4 on the affine covering of X , we see that $\mathcal{O}_{X'}$ is a flat $f^{-1}\mathcal{O}_{Y'}$ -module if and only if \mathcal{O}_X is a flat $f^{-1}\mathcal{O}_Y$ -module and the natural morphism $f^*\mathcal{J} \rightarrow \mathcal{I}$ is an isomorphism.

The following theorem of Illusie describes under which conditions a given deformation tuple can be extended to a w -compatible deformation.

Theorem 3.2.14 ([Ill71, Chapitre III, Théorème 2.1.7]). *For any deformation tuple as above, there exists an obstruction $\sigma_X \in \text{Ext}^2(L_{X/Y}, \mathcal{I})$ whose vanishing is sufficient and necessary for the existence of a w -compatible deformation X' . If the obstruction vanishes, the set of liftings constitutes a torsor under $\text{Ext}^1(L_{X/Y}, \mathcal{I})$.*

As a direct corollary, we obtain:

Corollary 3.2.15. *Let $Y \rightarrow Y'$ be an S -extension of Y by \mathcal{J} , and let $X \rightarrow Y$ be a flat morphism. There exists an obstruction class in $\text{Ext}^2(L_{X/Y}, f^*\mathcal{J})$ whose vanishing is sufficient and necessary for existence of a flat lifting $X' \rightarrow Y'$. If the obstruction class vanishes, then the set of flat liftings constitutes a torsor under $\text{Ext}^1(L_{X/Y}, f^*\mathcal{J})$.*

Remark 3.2.16. The obstruction σ_X is given by the composition:

$$L_{X/Y} \xrightarrow{K_{X/Y/S}} Lf^*L_{Y/S}[1] \xrightarrow{Lf^*\delta[1]} Lf^*\mathcal{J}[2] \xrightarrow{H^0} f^*\mathcal{J}[2] \xrightarrow{w[2]} \mathcal{I}[2],$$

where $\delta \in \text{Ext}^1(L_{Y/S}, \mathcal{J})$ denotes the cohomology class associated to the extension Y' (cf. Thm 3.2.8).

Remark 3.2.17. In the algebraic setting, the above considerations can be described by the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & B' & \longrightarrow & B \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & J & \longrightarrow & A' & \longrightarrow & A \longrightarrow 0. \end{array}$$

In this context, the obstruction classes for existence of B' lie in $\text{Ext}^2(L_{B/A}, I)$.

3.2.3 Lifting morphisms of schemes

In this paragraph, we introduce the notion of an infinitesimal deformation of a scheme morphism. Let us suppose the following diagram of schemes over a base S is given

$$\begin{array}{ccc}
 X_1 & \longrightarrow & X'_1 \\
 \downarrow h_1 & & \downarrow \\
 Z_1 & \xrightarrow{\quad f \quad} & Z'_1 \\
 \downarrow f & \nearrow \tilde{f} & \downarrow \\
 X_2 & \longrightarrow & X'_2 \\
 \downarrow h_2 & \nearrow g & \downarrow \\
 Z_2 & \longrightarrow & Z'_2,
 \end{array}$$

where the horizontal arrows are S -extensions of square-zero by ideals $\mathcal{I}_1, \mathcal{J}_1, \mathcal{I}_2$ and \mathcal{J}_2 , respectively viewed from the top, and vertical arrows give compatible morphisms of extensions. We ask whether there exists a morphism $f': X'_1 \rightarrow X'_2$ compatible with a given morphism $f^* \mathcal{I}_2 \rightarrow \mathcal{I}_1$ such that the diagram becomes commutative. Since a first-order thickening does not change the underlying topological space of a scheme, the problem can be expressed as a question concerning morphisms of sheaves of rings on X_1

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{I}_1 & \longrightarrow & \mathcal{O}_{X'_1} & \longrightarrow & \mathcal{O}_{X_1} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & h_1^{-1} \mathcal{J}_1 & \xrightarrow{\quad \quad} & h_1^{-1} \mathcal{O}_{Z'_1} & \xrightarrow{\quad \quad} & h_1^{-1} \mathcal{O}_{Z_1} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & f^{-1} \mathcal{I}_2 & \longrightarrow & f^{-1} \mathcal{O}_{X'_2} & \longrightarrow & f^{-1} \mathcal{O}_{X_2} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & u^{-1} \mathcal{J}_2 & \longrightarrow & u^{-1} \mathcal{O}_{Z'_2} & \longrightarrow & u^{-1} \mathcal{O}_{Z_2} \longrightarrow 0,
 \end{array}$$

where $u = g \circ h_1 = h_2 \circ f$. In a slightly simplified situation, when $Z_1 = Z_2$ and $Z'_1 = Z'_2$, this can be expressed using the following object.

Definition 3.2.18 (Deformation of morphism tuple). A *deformation of morphism tuple* over a square-zero extension $Z \rightarrow Z'$ by a \mathcal{O}_Z -module \mathcal{J} consists of:

a) a diagram of schemes:

$$\begin{array}{ccc}
 X_1 & \xrightarrow{i} & X'_1 \\
 \downarrow f & & \downarrow \\
 X_2 & \xrightarrow{j} & X'_2 \\
 \downarrow g & & \downarrow g' \\
 Z & \longrightarrow & Z'
 \end{array}
 \begin{array}{c}
 \left. \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right\} h \\
 \left. \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \right\} h'
 \end{array}$$

where i_k is an Z' -extension of X_k by an \mathcal{O}_{X_k} -module \mathcal{I}_k .

- b) a homomorphism $w : f^*\mathcal{I}_2 \rightarrow \mathcal{I}_1$ satisfying the equality $w_h = w \circ f^*w_g$ where $w_g : g^*\mathcal{J} \rightarrow \mathcal{I}_2$ and $w_h : h^*\mathcal{J} \rightarrow \mathcal{I}_1$ are the natural maps induced by g' and h' , respectively.

We recall the following theorem describing obstruction classes to lifting of morphisms in the scheme theoretic setting.

Theorem 3.2.19 ([Ill71, Chapitre III, Proposition 2.2.4]). *For every deformation of morphism tuple as above, there exists an obstruction $\sigma_f \in \text{Ext}^1(f^*L_{X_2/Z}, \mathcal{I}_1)$ whose vanishing is sufficient and necessary for the existence of a lifting $f' : X'_1 \rightarrow X'_2$ inducing w on the level of ideals of X_1 and X_2 .*

Remark 3.2.20. The obstruction in the theorem above is given by the unique preimage under the natural mapping $\text{Ext}^1(f^*L_{X_2/Z}, \mathcal{I}_1) \rightarrow \text{Ext}^1(f^*L_{X_2/Z'}, \mathcal{I}_1)$ of the difference $w \circ f^*e_2 - e_1 \circ df$ of arrows in the diagram:

$$\begin{array}{ccc} f^*L_{X_2/Z'} & \xrightarrow{f^*e_2} & f^*\mathcal{I}_2 \\ \downarrow df & & \downarrow w \\ L_{X_1/Z'} & \xrightarrow{e_1} & \mathcal{I}_1, \end{array}$$

where $e_k \in \text{Ext}^1(L_{X_k/Z'}, \mathcal{I}_k) \simeq \text{Exal}_{Z'}(\mathcal{O}_{X_k}, \mathcal{I}_k)$ (cf. Theorem 3.2.8) are classes of Z' -extensions X'_k .

Remark 3.2.21. In the case when X'_1 and X'_2 are just flat extension over Z' , the obstruction lie in the group $\text{Ext}^1(L_{X_2/Z}, f^*\mathcal{J})$.

Remark 3.2.22. A purely algebraic analogue of the situation described above can be illustrated by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_2 & \longrightarrow & B'_2 & \longrightarrow & B_2 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \xrightarrow{\nu} & I_2 & \xrightarrow{\quad} & A'_2 & \xrightarrow{\quad} & A_2 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & N_1 & \longrightarrow & B'_1 & \longrightarrow & B_1 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I_1 & \longrightarrow & A'_1 & \longrightarrow & A_1 \longrightarrow 0. \end{array}$$

In this context, the obstruction classes for existence of the dashed arrow lies in $\text{Ext}^1_{B_1}(L_{B_1/A_1}, N_2) = \text{Ext}^1_{B_2}(L_{B_1/A_1} \otimes_{B_1}^L B_2, N_2)$.

3.3 Naive cotangent complex and explicit obstructions for lifting morphisms

In this paragraph, we introduce the naive cotangent complex and then present how it is applied to compute obstructions for lifting morphisms explicitly. We focus on the case of ring homomorphisms, or equivalently morphisms of affine schemes. Our presentation is based on [Sta17, Tag 00S0]. Let $A \rightarrow B$ be a ring homomorphism. It turns out that the complex $\tau_{\geq -1}L_{B/A}$ can be efficiently computed using the notion of so-called naive cotangent complex

Definition 3.3.1 (Naive cotangent complex). The *naive cotangent complex* $NL_{B/A}$ of a ring map $A \rightarrow B$ is the complex:

$$NL_{B/A} : I/I^2 \xrightarrow{f \mapsto df \otimes 1} \Omega_{A[B]}^1 \otimes_{A[B]} B$$

supported in degrees $[-1, 0]$, where I denotes the kernel of the evaluation map $\text{ev}_{B/A} : A[B] \rightarrow B$.

We have the following comparison result.

Theorem 3.3.2 ([Sta17, Tag 08RB]). *Let $A \rightarrow B$ be a ring map. Then $NL_{B/A}$ is canonically quasi-isomorphic to $\tau_{\geq -1} L_{B/A}$.*

It turns out that up to homotopy $NL_{B/A}$ can be computed from any polynomial presentation of B (see [Sta17, Tag 00S0]). More precisely, for any surjective morphism $\varphi : A[E] \rightarrow B$ with kernel J the complex $NL_{B/A}$ is quasi-isomorphic to

$$NL_{\varphi} : J/J^2 \xrightarrow{f \mapsto df \otimes 1} \Omega_{A[E]}^1 \otimes_{A[E]} B.$$

We claim that the naive cotangent complex allows us to compute the obstructions for lifting morphisms explicitly. In order to show the claim, we take an instance of deformation of morphism problem described by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_2 & \longrightarrow & B'_2 & \xrightarrow{q_2} & B_2 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \xrightarrow{\nu} & I_2 & \xrightarrow{\quad} & A'_2 & \xrightarrow{\quad} & A_2 \longrightarrow 0 \\ & & \uparrow & \nearrow f' & \uparrow & \nearrow f & \\ 0 & \longrightarrow & N_1 & \longrightarrow & B'_1 & \xrightarrow{q_1} & B_1 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I_1 & \longrightarrow & A'_1 & \longrightarrow & A_1 \longrightarrow 0. \end{array}$$

We consider the naive cotangent complex NL_{φ} defined as

$$NL_{\varphi} : J/J^2 \xrightarrow{f \mapsto df \otimes 1} \Omega_{A_1[E]}^1 \otimes_{A_1[E]} B_1.$$

for a surjective homomorphism $\varphi : A_1[E] \rightarrow B_1$ with kernel J .

Theorem 3.3.3 ([Sta17, Tag 08S3]). *There exists an obstruction class $\sigma_{NL_{\varphi}} \in \text{Ext}_{B_1}^1(NL_{\varphi}, N_2)$ sufficient and necessary for existence of a morphism f' filling the above diagram.*

Proof. First, we observe that for any B_1 -module M the spectral sequence

$$E_1^{pq} = \text{Ext}_{B_1}^q(NL_{\varphi}^{-p}, M) \Rightarrow \text{Ext}_{B_1}^{p+q}(NL_{\varphi}, M)$$

arising from the trivial filtration of NL_{φ} yields an isomorphism

$$\text{Ext}_{B_1}^1(NL_{\varphi}, M) \simeq \text{Hom}_{B_1}(J/J^2, M) \Big/ \text{Hom}_{B_1}(\Omega_{A_1[E]/A_1}^1 \otimes_{A_1[E]} B_1, M).$$

Therefore it suffices to find the obstruction class in

$$\mathrm{Hom}_{B_1}(J/J^2, N_2) \Big/ \mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1}^1 \otimes_{A_1[E]} B_1, N_2).$$

Let $u: A'_1[E] \rightarrow B_1$ be the composition of the natural reduction map $A'_1[E] \rightarrow A_1[E]$ with the surjection $\phi: A_1[E] \rightarrow B_1$, and let J' be its kernel. We choose a lifting $A'_1[E] \rightarrow B'_1$ of u given by liftings in B'_1 of elements $\varphi(e)$, for $e \in E$. This yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J' & \longrightarrow & A'_1[E] & \xrightarrow{u} & B_1 \longrightarrow 0 \\ & & \downarrow w & & \downarrow & & \parallel \\ 0 & \longrightarrow & N_1 & \longrightarrow & B'_1 & \longrightarrow & B_1 \longrightarrow 0. \end{array}$$

From the commutativity of the diagram, we see that B'_1 is the pushout of mappings $w: J' \rightarrow N_1$ and $J' \rightarrow A'_1[E]$, and therefore to produce a mapping $B'_1 \rightarrow B'_2$ lifting $f: B_1 \rightarrow B_2$ and inducing $\nu: N_1 \rightarrow N_2$ on the level of kernels of the mappings $B'_i \rightarrow B_i$, we need to provide a morphism $q: A'_1[E] \rightarrow B'_2$ lifting $f \circ u$ and satisfying $\nu \circ w = q|_{J'}$. We choose an arbitrary morphism q lifting $f \circ u$ and consider the morphism $\nu \circ w - q|_{J'}$ which vanishes if and only if q descends to a homomorphism $B'_1 \rightarrow B'_2$. Since $q_2 \circ q|_{J'} = 0$, the morphism factors through N_2 . Moreover, by a simple computation, we see that it vanishes on $I_1 A'_1[E]$ and on J'^2 and hence gives an element $\tilde{\sigma}_q \in \mathrm{Hom}_{B_1}(J/J^2, N_2)$ defined unambiguously by the formula $\tilde{\sigma}_q(g) = \nu(w(\tilde{g})) - q(\tilde{g})$, where $\tilde{g} \in J'$ is a lifting of $g \in J$. To finish the proof, we observe that a different choice of a lifting $q: A'_1[E] \rightarrow B'_2$ shifts $\tilde{\sigma}_q$ by an element from $\mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1}^1 \otimes_{A_1[E]} B_1, N_2)$. Indeed, let $q': A'_1[E] \rightarrow B'_2$ and $q: A'_1[E] \rightarrow B'_2$ be two liftings of $f \circ u$. Then for any $g \in J$ we have

$$\tilde{\sigma}_q(g) - \tilde{\sigma}_{q'}(g) = (q' - q)(\tilde{g}) = \sum_{e \in E} (q - q')(e) \cdot q \left(\frac{\partial \tilde{g}}{\partial e} \right) = \sum_{e \in E} (q - q')(e) \cdot f \left(\frac{\partial g}{\partial e} \right),$$

since $q(e) - q'(e) \in N_2$ and therefore the products $(q - q')(e) \cdot q \left(\frac{\partial \tilde{g}}{\partial e} \right)$ depend solely on the residue $f \left(\frac{\partial g}{\partial e} \right)$. This means that $\tilde{\sigma}_q - \tilde{\sigma}_{q'}$ is an image under $\mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1}^1 \otimes_{A_1[E]} B_1, N_2) \rightarrow \mathrm{Hom}_{B_1}(J/J^2, N_2)$ of the homomorphism defined by the formula $de \mapsto q(e) - q'(e)$, for every $e \in E$. □

3.4 Functoriality of obstruction classes

3.4.1 Lifting schemes

Liftability obstructions satisfy the following functoriality property (for the sake of convenience we assume that the ideals of thickenings are flat).

Definition 3.4.1 (Morphism of deformation tuples). *A morphism of deformation tuples*

over a base S is denoted by a diagram

$$\left[\begin{array}{c} X_1 \\ \downarrow f_1 \\ Y_1 \xrightarrow{i} Y'_1 \\ \downarrow \\ S, \end{array} \right] \xrightarrow[\text{deformation tuples}]{\text{a morphism of}} \left[\begin{array}{c} X_2 \\ \downarrow f_2 \\ Y_2 \xrightarrow{i_2} Y'_2 \\ \downarrow \\ S \end{array} \right] \\ \left[(\mathcal{I}_1, \mathcal{J}_1, w_1 : f_1^* \mathcal{J}_1 \rightarrow \mathcal{I}_1) \right] \quad \left[(\mathcal{I}_2, \mathcal{J}_2, w_2 : f_2^* \mathcal{J}_2 \rightarrow \mathcal{I}_2), \right]$$

and consists of:

a) a diagram of schemes over S :

$$\begin{array}{ccccc} & & X_1 & & \\ & \swarrow g & \downarrow f_1 & & \\ X_2 & & Y_1 & \xrightarrow{i_1} & Y'_1 \\ & \searrow f_2 & \swarrow h & & \swarrow h' \\ Y_2 & \xrightarrow{i_2} & Y'_2, & & \end{array}$$

where h' is a morphism inducing an \mathcal{O}_{Y_1} -module homomorphism $u : h^* \mathcal{J}_2 \rightarrow \mathcal{J}_1$,

b) a homomorphism $v : g^* \mathcal{I}_2 \rightarrow \mathcal{I}_1$ fitting into a commutative diagram:

$$\begin{array}{ccc} g^* f_2^* \mathcal{J}_2 \simeq f_1^* h^* \mathcal{J}_2 & \xrightarrow{g^* w_2} & g^* \mathcal{I}_2 \\ \downarrow f^* u & & \downarrow v \\ f^* \mathcal{J}_1 & \xrightarrow{w_1} & \mathcal{I}_1. \end{array}$$

Lemma 3.4.2 (Functoriality of obstructions to lifting schemes). *For any morphism of deformation tuples as above, the obstruction classes to existence of liftings fit in the commutative diagrams:*

$$\begin{array}{ccc} Lg^* L_{X_2/Y_2} \xrightarrow{g^* \sigma_{X_2}} Lg^* \mathcal{I}_2 & & L_{X_2/Y_2} \xrightarrow{\sigma_{X_2}} \mathcal{I}[2] \\ \downarrow dg & & \downarrow dg \\ L_{X_1/Y_1} \xrightarrow{\sigma_{X_1}} \mathcal{I}, & & Rg_* L_{X_1/Y_1} \xrightarrow{Rg_* \sigma_{X_1}} Rg_* \mathcal{I}[2]. \end{array}$$

Proof. We use commutativity of the diagram:

$$\begin{array}{ccccccc} & & & & Lg^* \sigma_{X_2} & & \\ & & & & \curvearrowright & & \\ Lg^* L_{X_2/Y_2} & \xrightarrow{g^* K_{X_2/Y_2/S}} & Lg^* Lf_2^* L_{Y_2/S}[1] \simeq Lf^* Lh^* L_{Y_2/S}[1] & \longrightarrow & Lg^* Lf'^* \mathcal{J}_2[2] & \xrightarrow{Lg^* H^0} & Lg^* f_2^* \mathcal{J}_2[2] \xrightarrow{Lg^* w_2} Lg^* \mathcal{I}_2 \\ \downarrow dg & & \downarrow Lf^* di & & \downarrow f^* u & & \downarrow v \circ H^0 \\ L_{X_1/Y_1} & \xrightarrow{K_{X_1/Y_1/S}} & Lf^* L_{Y_1/S}[1] & \longrightarrow & Lf^* \mathcal{J}_1[2] & \xrightarrow{H^0} & f^* \mathcal{J}_1[2] \xrightarrow{w} \mathcal{I}_1, \\ & & & & \sigma_{X_1} & & \end{array}$$

which follows from:

- a) the description of obstructions classes,
- b) functoriality of Kodaira–Spencer class (left–most square),
- c) commutativity of the lower part of the diagram in the definition of morphism of deformation tuples (middle square),
- d) the assumption on the homomorphisms of flat ideals (right–most square).

We consequently derive the following diagram which is constituted by boundary arrows.

$$\begin{array}{ccc}
Lg^*L_{X_2/Y_2} & \xrightarrow{g^*\sigma_{X_2}} & Lg^*\mathcal{I}_2 \\
\downarrow dg & & \downarrow v \\
L_{X_1/Y_1} & \xrightarrow{\sigma_{X_1}} & \mathcal{I}_1.
\end{array}$$

After application of Rg_* and the natural adjunction this yields the commutativity of:

$$\begin{array}{ccc}
L_{X_2/Y_2} & \xrightarrow{\sigma_{X_2}} & \mathcal{I}_2[2] \\
\downarrow \eta_{L_{X_2/Y_2}} & & \downarrow \eta_{\mathcal{I}_2[2]} \\
Rg_*Lg^*L_{X_2/Y_2} & \xrightarrow{Rg_*Lg^*\sigma_{X_2}} & Rg_*Lg^*\mathcal{I}_2[2] \\
\downarrow Rg_*dg & & \downarrow Rg_*v \\
Rg_*L_{X_1/Y_1} & \xrightarrow{Rg_*\sigma_{X_1}} & Rg_*\mathcal{I}_1[2],
\end{array}$$

where $\eta_{\mathcal{A}}$ is the co-unit of the adjunction for an object \mathcal{A} . This gives the desired result. \square

3.4.2 Lifting morphisms

Here, we analyse the functoriality properties of obstruction classes for lifting morphisms.

Definition 3.4.3 (Morphism of deformation of morphism tuples). A *morphism of deformation of morphism tuples* over a square-zero extension $S \rightarrow \tilde{S}$ by an \mathcal{O}_S -module is denoted by a diagram:

$$\left[\begin{array}{ccc}
X_2 & \xrightarrow{i_2} & X'_2 \\
\downarrow f_2 & & \downarrow \\
Y_2 & \xrightarrow{j_2} & Y'_2 \\
\downarrow g_2 & & \downarrow g'_2 \\
Z & \xrightarrow{\quad} & Z'
\end{array} \right] \xrightarrow[\text{of morphism tuples}]{\text{a morphism of deformation}} \left[\begin{array}{ccc}
X_1 & \xrightarrow{i_1} & X' \\
\downarrow f_1 & & \downarrow \\
Y_1 & \xrightarrow{j_1} & Y' \\
\downarrow g_1 & & \downarrow g'_1 \\
Z & \xrightarrow{\quad} & Z'
\end{array} \right]$$

$(\mathcal{I}_2, \mathcal{J}_2, w_2 : f_2^*\mathcal{J}_2 \rightarrow \mathcal{I}_2)$
 $\qquad\qquad\qquad$
 $(\mathcal{I}_1, \mathcal{J}_1, w : f^*\mathcal{J}_1 \rightarrow \mathcal{I}_1)$

and consists of a diagram of schemes

$$\begin{array}{ccccc}
& & X_2 & \xrightarrow{\quad} & X'_2 \\
& \swarrow k & \downarrow f_2 & \swarrow k' & \downarrow \text{dashed} \\
X_1 & \xrightarrow{\quad} & X'_1 & & \\
\downarrow f_1 & & \downarrow & & \downarrow \text{dashed} \\
& & Y_2 & \xrightarrow{\quad} & Y'_2 \\
& \swarrow l & \downarrow & \swarrow l' & \downarrow \text{dashed} \\
Y_1 & \xrightarrow{\quad} & Y'_1 & & \\
\downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{\quad} & S' & &
\end{array}$$

such that the mapping $q_k : k^* \mathcal{I}_1 \rightarrow \mathcal{I}_2$ and $q_l : l^* \mathcal{J}_1 \rightarrow \mathcal{J}_2$ fit into a commutative diagram:

$$\begin{array}{ccc}
k^* f_1^* \mathcal{J} \simeq f_2^* l^* \mathcal{J} & \xrightarrow{f_2^* q_l} & f_2^* \mathcal{J}_2 \\
\downarrow k^* w_1 & & \downarrow w_2 \\
k^* \mathcal{I}_1 & \xrightarrow{q_k} & \mathcal{I}_2.
\end{array}$$

We do not need this result in its full generality and therefore for the notational convenience we only state the case of flat extensions over a given S' .

Lemma 3.4.4 (Functoriality of obstructions to lifting morphisms). *For any morphism of deformation of morphism tuples as above where the associated ideals are flat, the obstructions σ_{f_1} and σ_{f_2} satisfy the relation:*

$$q_k \circ k^* \sigma_{f_1} = \sigma_{f_2} \circ f_2^* dl \circ u^{-1},$$

where $u : f_2^* l^* L_{X_2/S'} \rightarrow k^* f_1^* L_{X_2/S'}$ is the canonical isomorphism.

Proof. We observe that the obstructions fit into the following diagram:

$$\begin{array}{ccccccc}
& & & & f_2^* l^* e_{j_1} & & \\
& & & & \searrow & & \\
f_2^* l^* L_{Y_1/S'} & \xrightarrow[\simeq]{u} & k^* f_1^* L_{Y_1/S'} & \xrightarrow{k^* f_1^* e_{j_1}} & k^* f_1^* \mathcal{J}_1 & \xrightarrow[\simeq]{} & f_2^* l^* \mathcal{J}_1 \\
\downarrow f_2^* dl & & \downarrow k^* df_1 & & \downarrow k^* w_1 & & \downarrow f_2^* q_l \\
f_2^* L_{Y_2/S'} & & k^* L_{X_1/S'} & \xrightarrow{k^* e_{i_1}} & k^* \mathcal{I}_1 & & f_2^* \mathcal{J}_2 \\
& \searrow df_2 & \downarrow & & \downarrow q_k & \swarrow w_2 & \\
& & L_{X_2/S'} & \xrightarrow{e_{i_2}} & \mathcal{I}_2, & &
\end{array}$$

where the middle square is the pullback of σ_{f_1} along k and the bottom boundary mappings give rise to the obstruction σ_{f_2} . The functoriality is a consequence of the following statements

a) the description of obstructions classes,

- b) commutativity of the left square coming from transitivity of differentials for $f_1k = lf_2$,
- c) commutativity of the right square coming from the definition of morphism of deformations of morphism tuples,
- d) commutativity of the lower and boundary upper square which in fact constitute the obstructions for existence of k' and l' , respectively.
- e) flatness of the ideals, which allows us to omit derived pull-backs.

□

3.5 Deformations of affine schemes

In this section we present a few results concerning deformation theory of affine schemes. In particular, we prove that one can check whether there exists a $W_2(k)$ -lifting étale locally.

Lemma 3.5.1. *Let $X \simeq \text{Spec}(A)$ be an affine scheme essentially of finite type over k . Assume there exists an étale surjective covering $p : U \rightarrow X$ such that U is $W_2(k)$ -liftable (respectively F -liftable). Then, X is $W_2(k)$ -liftable (respectively F -liftable).*

Proof. We treat only the case of $W_2(k)$ -liftable as the proof in the case of F -liftable is analogous. By taking an affine Zariski covering of U we may assume that $U \simeq \text{Spec}(B)$. Let $\sigma_A \in \text{Ext}^2(L_{A/k}, A)$ and $\sigma_B \in \text{Ext}^2(L_{B/k}, B)$ be the obstruction classes to the existence of a $W_2(k)$ -lifting of $\text{Spec}(A)$ and $\text{Spec}(B)$, respectively. By Lemma 3.4.2 and the properties of the morphism p we see that $\text{Ext}^2(L_{B/k}, B) \simeq \text{Ext}^2(L_{A/k}, A) \otimes_A B$ and $\sigma_B = \sigma_A \otimes 1$ under this isomorphism. Therefore, the claim of the lemma follows from the fact that p is étale and surjective and hence faithfully flat. □

Along the same lines we can prove the following.

Lemma 3.5.2. *Let X be an affine scheme locally of finite type over k . Assume that for every closed point $x \in X$ the scheme $\text{Spec}(\mathcal{O}_{X,x})$ is $W_2(k)$ -liftable (respectively F -liftable). Then, X is $W_2(k)$ -liftable (respectively F -liftable).*

3.6 Functors of Artin rings and their applications

Here, we recall the formalism of functors of Artin rings. The classical reference for the topic is [Sch68]. The results concerning obstruction theories are well-explained in [FM98].

Suppose k be a field, Λ a complete local ring and let $\mathbf{Art}_\Lambda(k)$ denote the category of local Artinian Λ -algebras with residue field k . We consider covariant functors $\mathcal{F} : \mathbf{Art}_\Lambda(k) \rightarrow \mathbf{Sets}$, which naturally represent infinitesimal deformations of algebraic and geometric objects. The cases we are mostly interested in are $\Lambda = W_2(k)$ or $\Lambda = W(k)$ where k is a perfect field of positive characteristic p .

Definition 3.6.1 (Functor of Artin rings). A *functor of Artin rings* is a covariant functor $\mathcal{F} : \mathbf{Art}_\Lambda(k) \rightarrow \mathbf{Sets}$ such that $\mathcal{F}(k)$ consists of a single element. All such functors form a category with natural transformations as morphisms.

Remark 3.6.2. We assume that the considered functors satisfy additional property

$$\mathcal{F}(k[V] \times_k k[W]) \simeq \mathcal{F}(k[V]) \times \mathcal{F}(k[W]).$$

This implies that $\mathcal{F}(k[\varepsilon]/\varepsilon^2)$ admits a structure of a k -vector space (see [Sch68, Lemma 2.10]). The assumption is satisfied for every functor we consider in what follows.

3.6.1 Deformation functors and their morphisms

Definition 3.6.3 (Smooth morphism of functors). A natural transformation $\xi: \mathcal{F} \rightarrow \mathcal{G}$ of functors of Artin rings is *smooth* if for every surjective morphism $B \rightarrow A$ in $\mathbf{Art}_\Lambda(k)$ the natural mapping:

$$\mathcal{F}(B) \rightarrow \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(B)$$

is surjective.

From the special case $A = k$, we see that a smooth morphism is surjective.

Definition 3.6.4 (Tangent space and morphism). Suppose $\mathcal{F}: \mathbf{Art}_\Lambda(k) \rightarrow \mathbf{Sets}$ is a functor of Artin rings. The set $T_{\mathcal{F}} \stackrel{\text{def}}{=} \mathcal{F}(k[\varepsilon])$ is called the *tangent space* of a functor \mathcal{F} . For any morphism of deformation functors $\psi: \mathcal{F} \rightarrow \mathcal{G}$ the mapping $T_{\psi} \stackrel{\text{def}}{=} \psi_{k[\varepsilon]}: T_{\mathcal{F}} = \mathcal{F}(k[\varepsilon]) \rightarrow \mathcal{G}(k[\varepsilon]) = T_{\mathcal{G}}$ is called the *tangent map* of ψ .

Under certain condition (see [FM98]), satisfied in any of our applications, the tangent space admits a natural structure of a k -vector space such that the tangent morphism becomes k -linear.

Definition 3.6.5 (Obstruction theory). Suppose $\mathcal{F}: \mathbf{Art}_\Lambda(k) \rightarrow \mathbf{Sets}$ is a functor of Artin rings. An *obstruction theory* for \mathcal{F} is a pair $(V, \{\nu_e\}_{e \in \mathbf{SmallExt}(\Lambda)})$ of a vector space V and a family of mappings $\nu_e: \mathcal{F}(A) \rightarrow V \otimes_k I$ parametrised by infinitesimal extensions $e: 0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ and satisfying the following properties:

- i) (functoriality) for any morphism of small extensions

$$\begin{array}{ccccccc} e: 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow f_2 & & \downarrow f_1 \\ e': 0 & \longrightarrow & I' & \longrightarrow & B' & \longrightarrow & A' \longrightarrow 0, \end{array}$$

we have $(\text{id}_V \otimes \phi) \circ \nu_e = \nu_{e'} \circ \mathcal{F}(f_1)$.

- ii) (completeness) for any infinitesimal extension $e: 0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ and an element $a \in \mathcal{F}(A)$ the condition $\nu_e(a) = 0$ is equivalent to existence of $b \in \mathcal{F}(B)$ lifting a .

The choice of obstruction theory is not unique, e.g., any proper inclusion $i: V \rightarrow V'$ gives rise to a different obstruction theory $(V', i \circ \nu_e)$.

We now present a criterion for smoothness of morphism of functors in terms of their tangent and obstruction spaces. Let $(\mathcal{F}, (\text{Ob}_{\mathcal{F}}, \nu_e^{\mathcal{F}}))$ and $(\mathcal{G}, (\text{Ob}_{\mathcal{G}}, \nu_e^{\mathcal{G}}))$ be deformation functors together with associated obstruction theories and $\psi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism. We say that a linear mapping $c: \text{Ob}_{\mathcal{F}} \rightarrow \text{Ob}_{\mathcal{G}}$ is an *obstruction map* of ψ if for every infinitesimal extension $e: 0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ we have $\nu_e^{\mathcal{G}} \circ \psi_A = (c \otimes_k \text{id}_I) \circ \nu_e^{\mathcal{F}}$. We emphasize that the notion of obstruction map of ψ does not depend solely on ψ but also on the choice of obstruction theories for \mathcal{F} and \mathcal{G} .

We have the following criterion of Fantechi and Manetti.

Lemma 3.6.6 ([FM98, Lemma 6.1]). *Suppose $(\mathcal{F}, (\text{Ob}_{\mathcal{F}}, \nu_{\mathcal{F}}^{\mathcal{F}}))$ and $(\mathcal{G}, (\text{Ob}_{\mathcal{G}}, \nu_{\mathcal{G}}^{\mathcal{G}}))$ are deformation functors together with associated obstruction theories. Let $\psi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of functors admitting an obstruction map $\text{Ob}_{\psi} : \text{Ob}_{\mathcal{F}} \rightarrow \text{Ob}_{\mathcal{G}}$. Then, ψ is smooth if the following conditions hold:*

- i) $T_{\psi} : T_{\mathcal{F}} \rightarrow T_{\mathcal{G}}$ is surjective,
- ii) $\text{Ob}_{\psi} : \text{Ob}_{\mathcal{F}} \rightarrow \text{Ob}_{\mathcal{G}}$ is injective.

3.6.2 Compatible systems of deformations

Suppose X is a scheme over a field k . By Def_X we mean the deformation functor of X , that is, a covariant functor from the category $\mathbf{Art}_{W(k)}(k)$ of Artinian local $W(k)$ -algebras with residue field k to the category of sets defined by the formula:

$$\mathbf{Art}_{W(k)}(k) \ni (A, \mathfrak{m}_A) \mapsto \text{Def}_X(A) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \text{isomorphism classes of flat} \\ \text{deformations of } X \text{ over } \text{Spec}(A) \end{array} \right\}.$$

Morphism of deformations $(\mathcal{X}, \phi_{\mathcal{X}}) \rightarrow (\mathcal{X}', \phi_{\mathcal{X}'})$ is a morphism of schemes $\mathcal{X} \rightarrow \mathcal{X}'$ such that its restriction to the special fibre commutes with isomorphisms ϕ . It turns out that the tangent space of Def_X can be described, and there exists a natural obstruction theory (cf. Definition 3.6.5). More precisely, we have the following standard result.

Proposition 3.6.7 ([Har10, Theorem 10.2]). *The tangent space T_{Def_X} is given by $\text{Ext}^1(L_{X/k}, \mathcal{O}_X)$. There exists an obstruction theory with the obstruction space given by $\text{Ext}^2(L_{X/k}, \mathcal{O}_X)$.*

Proof. Both assertions follow from Corollary 3.2.15. For the first one, we observe that according to the corollary the tangent space $T_{\text{Def}_X} = \text{Def}_X(k[\varepsilon]/\varepsilon^2)$ is a torsor under $\text{Ext}^1(L_{X/k}, \mathcal{O}_X)$. For the second, we consider a surjective homomorphism $A' \rightarrow A$ in $\mathbf{Art}_{W(k)}(k)$ with kernel I , satisfying $\mathfrak{m}_{A'}I = 0$. Using the corollary, we see that for every element $\mathcal{X} \in \text{Def}_X(A)$ the obstruction class to existence of an A' -lifting of \mathcal{X} lies in $\text{Ext}^1(L_{X/k}, f^*I) \simeq \text{Ext}^1(L_{X/k}, \mathcal{O}_X) \otimes_k I$. This means that $\text{Ext}^2(L_{X/k}, \mathcal{O}_X)$ (together with the natural obstruction map) gives an obstruction theory for Def_X . \square

The above setting can be slightly generalized.

Remark 3.6.8. Let $Z = \{Z_i\}_{i \in I}$ be a family of closed subschemes of X indexed by a preorder I (i.e., a set with a reflexive and transitive binary relation), such that Z_i is a closed subscheme of Z_j whenever $i \leq j$ (in other words, I is a small category whose morphism sets have at most one element, and Z is a functor from I to the category of closed subschemes of X). We denote by $\text{Def}_{X,Z}$ the functor of flat deformations of X together with compatible embedded deformations of the Z_i , preserving the inclusion relations given by the relation \leq . If $f : X \rightarrow Y$ is a map of k -schemes, we denote by Def_f the functor of flat deformations of X , and Y along with a deformation of f .

One of our main tools is the following proposition, which one can prove along the same lines as [LS14, Proposition 2.2]. See also [CvS09] and [Wah79], where this idea appeared previously.

Proposition 3.6.9. *1) Let $f : Y \rightarrow X$ be a map satisfying $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. Then there exists a natural transformation $\text{Def}_Y \rightarrow \text{Def}_X$. More generally, if $W = \{W_i\}_{i \in I}$ (resp. $Z = \{Z_i\}_{i \in I}$) is a family of closed subschemes of Y (resp. X) parametrized by a preorder I , and if $Rf_*\mathcal{O}_{W_i} = \mathcal{O}_{Z_i}$ (in particular, $Z_i = f(W_i)$), then there exists a natural transformation $\text{Def}_{Y,W} \rightarrow \text{Def}_{X,Z}$.*

2) Let X be a smooth scheme, $Z \subseteq X$ a smooth closed subscheme of codimension ≥ 2 , $f : Y = \text{Bl}_Z X \rightarrow X$ the blow-up of X along Z , $E = \{E_j\}_{j \in J}$ the set of connected components of $f^{-1}(Z)$. Then the forgetful transformation $\text{Def}_{Y,E} \rightarrow \text{Def}_Y$ is an isomorphism (here the index set J is given the trivial order). More generally, if $W = \{W_i\}_{i \in I}$ is a family of closed subschemes of Y , then the forgetful transformation $\text{Def}_{Y,W \sqcup E} \rightarrow \text{Def}_{Y,W}$ is an isomorphism. Here by $W \sqcup E$ we mean the family $\{W_i\}_{i \in I} \sqcup \{E_j\}_{j \in J}$ parametrized by $I \sqcup J$ with no non-trivial relations between I and J .

3.6.3 Embedded deformations $\text{Hilb}_{X,\mathbb{P}^n}$ and Def_X

Example 3.6.10 (Embedded deformations of a projective scheme). Suppose $X \subset \mathbb{P}_k^n$ is a closed subscheme. The functor of *embedded deformations* $\text{Hilb}_{X,\mathbb{P}^n} : \mathbf{Art}_\Lambda(k) \rightarrow \mathbf{Sets}$ is defined by the assignment:

$$\mathbf{Art}_\Lambda(k) \ni A \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of closed subschemes of } \mathbb{P}_A^n \\ \text{flat over } \text{Spec}(A) \text{ and restricting to } X \text{ over } \text{Spec}(k) \end{array} \right\}.$$

The following proposition describes basic properties of $\text{Hilb}_{X,\mathbb{P}^n}$.

Proposition 3.6.11 ([Har10, Theorem 1.1]). *The tangent space of $\text{Hilb}_{X,\mathbb{P}^n}$ is naturally isomorphic to $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$. Furthermore, it admits an obstruction theory with the obstruction space equal to $H^1(X, \mathcal{N}_{X/\mathbb{P}^n})$.*

We now present an exemplary application of Lemma 3.6.6.

Example 3.6.12. For any closed embedding $X \subset \mathbb{P}^n$ of a smooth scheme X/k , we obtain a natural morphism of functors $\psi : \text{Hilb}_{X,\mathbb{P}^n} \rightarrow \text{Def}_X$, admitting an obstruction morphism, defined by taking the underlying abstract deformation of an embedded deformation. Its tangent and obstruction mappings:

$$\begin{aligned} T_\psi : T_{\text{Hilb}_{X,\mathbb{P}^n}} &= H^0(X, \mathcal{N}_{X/\mathbb{P}^n}) \rightarrow H^1(X, \mathcal{T}_X) \simeq \text{Ext}^1(L_{X/k}, \mathcal{O}_X) = T_{\text{Def}_X} \\ \text{Ob}_\psi : \text{Ob}_{\text{Hilb}_{X,\mathbb{P}^n}} &= H^1(X, \mathcal{N}_{X/\mathbb{P}^n}) \rightarrow H^2(X, \mathcal{T}_X) \simeq \text{Ext}^2(L_{X/k}, \mathcal{O}_X) = \text{Ob}_{\text{Def}_X} \end{aligned}$$

are given by the canonical morphisms coming from the long exact sequence of cohomology associated to:

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^n|X} \longrightarrow \mathcal{N}_{X/\mathbb{P}^n} \longrightarrow 0.$$

As a simple corollary of Lemma 3.6.6 we have the following lemma applicable for example to projective Calabi–Yau varieties of $\dim X \geq 3$.

Lemma 3.6.13. *Suppose X is a projective scheme satisfying $H^2(X, \mathcal{O}_X) = 0$. For sufficiently positive embedding of $X \subset \mathbb{P}^N$ the morphism $\psi : \text{Hilb}_{X,\mathbb{P}^N} \rightarrow \text{Def}_X$ is smooth. In particular, every abstract deformation arises as an embedded one.*

Proof. By Lemma 3.6.6 we need to show that $H^0(X, \mathcal{N}_{X/\mathbb{P}^N}) \rightarrow H^1(X, \mathcal{T}_X)$ is surjective and $H^1(X, \mathcal{N}_{X/\mathbb{P}^N}) \rightarrow H^2(X, \mathcal{T}_X)$ is injective for sufficiently ample embedding $X \rightarrow \mathbb{P}^N$. By the long exact sequence of cohomology it suffices to show that $H^1(X, \mathcal{T}_{\mathbb{P}^N|X}) = 0$. By restricting the Euler sequence to X we obtain:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^N|X} \longrightarrow 0.$$

By another long exact sequence of cohomology we see that:

$$\cdots \longrightarrow H^1(X, \mathcal{O}_X(1)^{\oplus n+1}) \longrightarrow H^1(X, \mathcal{T}_{\mathbb{P}^N|X}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X(1)^{\oplus n+1}) \longrightarrow \cdots,$$

which gives the claim by the Serre's vanishing, i.e., for any coherent sheaf \mathcal{F} on X we have $H^1(X, \mathcal{F}(1)) = H^2(X, \mathcal{F}(1)) = 0$ if the embedding $X \rightarrow \mathbb{P}^N$ is sufficiently ample. \square

Remark 3.6.14. In particular, for sufficiently positive Veronese embedding of a smooth scheme X satisfying $H^2(X, \mathcal{O}_X) = 0$, the mapping of functors $\psi : \text{Hilb}_{X, \mathbb{P}^N} \rightarrow \text{Def}_X$ is smooth.

3.6.4 Deformations of open subsets

We proceed with a classical lemma comparing abstract deformation functors of a scheme X and its open subset U . We give a modern formulation together with a proof based on the cotangent complex.

Lemma 3.6.15 ([Art76, Proposition 9.2]). *Let $j : U \rightarrow X$ be the inclusion of an open subset. Assume, that X satisfies property S_3 at any point \mathfrak{p} of the complement $Z = X \setminus U$. Then, the natural morphism of deformation functors $\chi_j : \text{Def}_X \rightarrow \text{Def}_U$ coming from the restriction is smooth.*

Proof. By the long exact sequence of $\text{Ext}^\bullet(L_{X/k}, -)$ groups associated to a distinguished triangle (cf. Lemma 2.4.8):

$$\mathcal{O}_X \longrightarrow Rj_*\mathcal{O}_U \longrightarrow R\underline{\Gamma}_Z(X, \mathcal{O}_X)[1] \longrightarrow \mathcal{O}_X[1],$$

and the identification $\text{Ext}^i(L_{X/k}, Rj_*\mathcal{O}_U) \simeq \text{Ext}^i(L_{U/k}, \mathcal{O}_U)$, for every $i \in \mathbb{N}$, we obtain a sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & \text{Ext}^0(L_{X/k}, R\underline{\Gamma}_Z(\mathcal{O}_X)[1]) & \longrightarrow & \text{Ext}^1(L_{X/k}, \mathcal{O}_X) & \xrightarrow{t_j} & \text{Ext}^1(L_{U/k}, \mathcal{O}_U) \\ & & & & & \searrow & \\ & & & & & & \text{Ext}^1(L_{X/k}, R\underline{\Gamma}_Z(\mathcal{O}_X)[1]) \longrightarrow \text{Ext}^2(L_{X/k}, \mathcal{O}_X) \xrightarrow{o_j} \text{Ext}^2(L_{U/k}, \mathcal{O}_U) \end{array}$$

where the mappings t_j and o_j can be identified with T_{χ_j} and Ob_{χ_j} . Therefore, by Lemma 3.6.6 it suffices to prove that $\text{Ext}^1(L_{X/k}, R\underline{\Gamma}_Z(\mathcal{O}_X)[1]) = 0$. This follows directly using the spectral sequence

$$\text{Ext}^p(L_{X/k}, \mathcal{H}_Z^{q+1}(X, \mathcal{O}_X)) = \text{Ext}^p(L_{X/k}, \mathcal{H}^q(R\underline{\Gamma}_Z(\mathcal{O}_X)[1])) \Rightarrow \text{Ext}^{p+q}(L_{X/k}, R\underline{\Gamma}_Z(\mathcal{O}_X)[1])$$

and Lemma 2.4.24. The proof is thus finished. \square

3.6.5 Deformations of cones over projective schemes

We now proceed to the summary of deformation theory of cones. Again, X denotes a closed subscheme of \mathbb{P}^n and we assume that $C \stackrel{\text{def}}{=} \text{Cone}_{X, \mathbb{P}^n}$ is normal at its vertex (cf. §2.5.1), that is, $\text{Cone}_{X, \mathbb{P}^n} \simeq \text{Cone}_{X, \mathcal{O}_X(1)}$. For a thorough exposition (in fact equicharacteristic) expressed in classical terms of tangent sheaves one may take a look at [Art76, Section 11 and 12]. We begin with a proposition.

Proposition 3.6.16 ([Art76, Theorem 12.1, Lemma 12.1]). *There exists a morphism of deformation functors $\phi : \text{Hilb}_{X, \mathbb{P}^n} \xrightarrow{\phi} \text{Def}_C$ defined by performing cone construction relatively. The tangent and obstruction mappings of ϕ satisfy the properties:*

- (1) The tangent mapping $T_\phi : T_{\text{Hilb}_{X, \mathbb{P}^n}} \rightarrow T_{\text{Def}_C}$ can be identified with the canonical homomorphism

$$H^0(X, \mathcal{N}_{X/\mathbb{P}^n}) \rightarrow \text{Coker} \left(\bigoplus_{k \in \mathbb{Z}} H^0(X, \mathcal{T}_{\mathbb{P}^n|X}(k)) \rightarrow \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathcal{N}_{X/\mathbb{P}^n}(k)) \right),$$

coming from the long exact cohomology sequence associated to

$$0 \longrightarrow \bigoplus_{k \in \mathbb{Z}} \mathcal{T}_X(k) \longrightarrow \bigoplus_{k \in \mathbb{Z}} \mathcal{T}_{\mathbb{P}^n|X}(k) \longrightarrow \bigoplus_{k \in \mathbb{Z}} \mathcal{N}_{X/\mathbb{P}^n}(k) \longrightarrow 0,$$

- (2) The tangent mapping T_ϕ is surjective if $\bigoplus_{k \neq 0} H^1(X, \mathcal{T}_X(k)) = 0$.

- (3) The obstruction mapping Ob_ϕ is injective.

Remark 3.6.17. The identification of the tangent space $T_{\text{Def}_C} = \text{Ext}^1(L_{C/k}, \mathcal{O}_C)$ with the given cokernel follows from an explicit calculation of the long exact sequence of $\text{Ext}^\bullet(-, \mathcal{O}_C)$ groups for a distinguished triangle:

$$Li^* L_{\mathbb{A}^{n+1}/k} \longrightarrow L_{C/k} \longrightarrow L_{C/\mathbb{A}^{n+1}} \longrightarrow Li^* L_{\mathbb{A}^{n+1}/k}[1]$$

associated with the inclusion of schemes $i : C \rightarrow \mathbb{A}^{n+1}$.

As a corollary we obtain the following result comparing the deformation theory of a projective scheme X and the cone over its sufficiently large ample embedding.

Corollary 3.6.18. *Let X be a projective scheme. Then, there exists a sufficiently large Veronese embedding of $X \subset \mathbb{P}^N$ such that the morphism of functor $\phi_d : \text{Hilb}_{X, \mathbb{P}^N} \rightarrow \text{Def}_{\text{Cone}_{X, \mathbb{P}^N}}$ is smooth.*

Proof. Firstly, by Proposition 2.5.2 we see that by taking sufficiently large Veronese embedding we may assume that the cone is in normal. This allows us to apply Proposition 3.6.16. By part (2), we need to show that for sufficiently large d we have $\bigoplus_{k \neq 0} H^1(X, \mathcal{T}_X(kd)) = 0$. This in turn follows from Serre's vanishing and Serre duality. \square

By combining Lemma 3.6.13 with Corollary 3.6.18 we obtain:

Proposition 3.6.19. *For any smooth projective scheme X satisfying $H^2(X, \mathcal{O}_X) = 0$, and a sufficiently large Veronese embedding $X \subset \mathbb{P}^N$, the morphisms of deformation functors ϕ and ψ given in a diagram:*

$$\text{Def}_{\text{Cone}_{X, \mathbb{P}^N}} \xleftarrow{\phi} \text{Hilb}_{X, \mathbb{P}^N} \xrightarrow{\psi} \text{Def}_X,$$

are smooth.

3.6.6 Deformations of products

In this paragraph, we compare the deformation functor of the product of two schemes with the product of their deformation functors.

Proposition 3.6.20. *The morphism of deformation functors:*

$$\text{prod}_{X,Y} : \text{Def}_X \times \text{Def}_Y \rightarrow \text{Def}_{X \times Y}, \quad (\tilde{X}, \tilde{Y}) \mapsto \tilde{X} \times_{\text{Spec}(A)} \tilde{Y}$$

is smooth (in particular levelwise surjective) if $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0$.

Proof. By the above general considerations and the additivity of Kodaira–Spencer class we see that the morphisms on tangent and obstruction space

$$\begin{aligned} T_{\text{prod}_{X,Y}} : \text{Ext}^1(L_{X/k}, \mathcal{O}_X) \oplus \text{Ext}^1(L_{Y/k}, \mathcal{O}_Y) &\rightarrow \text{Ext}^1(Lp^*L_{X/k} \oplus Lq^*L_{Y/k}, \mathcal{O}_{X \times Y}), \\ \text{Ob}_{\text{prod}_{X,Y}} : \text{Ext}^2(L_{X/k}, \mathcal{O}_X) \oplus \text{Ext}^2(L_{Y/k}, \mathcal{O}_Y) &\rightarrow \text{Ext}^2(Lp^*L_{X/k} \oplus Lq^*L_{Y/k}, \mathcal{O}_{X \times Y}), \end{aligned}$$

are given as direct sums of morphisms:

$$\begin{aligned} \text{Ext}^\bullet(L_{X/k}, \mathcal{O}_X) &\rightarrow \text{Ext}^\bullet(L_{X/k}, Rp_*\mathcal{O}_{X \times Y}) \simeq \text{Ext}^\bullet(Lp^*L_{X/k}, \mathcal{O}_{X \times Y}); \\ \text{Ext}^\bullet(L_{Y/k}, \mathcal{O}_Y) &\rightarrow \text{Ext}^\bullet(L_{Y/k}, Rq_*\mathcal{O}_{X \times Y}) \simeq \text{Ext}^\bullet(Lq^*L_{Y/k}, \mathcal{O}_{X \times Y}), \end{aligned}$$

which arise from the natural distinguished triangles

$$\mathcal{O}_X \longrightarrow Rp_*\mathcal{O}_{X \times Y} \longrightarrow C_p \quad \text{and} \quad \mathcal{O}_Y \longrightarrow Rq_*\mathcal{O}_{X \times Y} \longrightarrow C_q,$$

induced by the structure morphisms $p^\#$ and $q^\#$ of the projections.

By the assumptions and the spectral sequence:

$$E_{ij}^2 = \text{Ext}^i(L_{X/k}, \mathcal{H}^j(C_p)) \Rightarrow \text{Ext}^{i+j}(L_{X/k}, C_p)$$

we see that $\text{Ext}^1(L_{X/k}, C_p) = H^1(Y, \mathcal{O}_Y) \otimes_k \text{Ext}^0(L_{X/k}, \mathcal{O}_X) = 0$. Analogously we obtain $\text{Ext}^1(L_{Y/k}, C_q) = 0$. Therefore $T_{\text{prod}_{X,Y}}$ is surjective and $\text{Ob}_{\text{prod}_{X,Y}}$ is injective, which by Lemma 3.6.6 implies that $\text{prod}_{X,Y}$ is a smooth morphism of deformation functors. \square

As a simple corollary we obtain:

Proposition 3.6.21. *Let $f : X \rightarrow Y$ be a morphism of schemes over a field k satisfying $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0$. If $\text{Bl}_{\Gamma_f}(X \times Y)$ lifts to $A \in \mathbf{Art}_{W(k)}(k)$, then there exist A -liftings of X and Y together with a lifting of f .*

Proof. Assume $\text{Bl}_{\Gamma_f}(X \times Y)$ lifts to A . By Proposition 3.6.9 there exists a deformation $\widetilde{X \times Y}$ of the product $X \times Y$ together with an embedded deformation $\widetilde{\Gamma}_f$ of Γ_f . By Proposition 3.6.20 the A -scheme $\widetilde{X \times Y}$ is isomorphic to $\widetilde{X} \times_{\text{Spec}(A)} \widetilde{Y}$ for some deformations of X and Y . The restriction of the projection $\widetilde{p}_X : \widetilde{X} \times_{\text{Spec}(A)} \widetilde{Y} \rightarrow \widetilde{X}$ to $\widetilde{\Gamma}_f$ is an isomorphism (as its restriction to $\text{Spec}(k)$ is an isomorphism) and therefore the tuple $(\widetilde{X}, \widetilde{Y}, \widetilde{p}_Y \circ (\widetilde{p}_X|_{\widetilde{\Gamma}_f})^{-1})$ gives the desired pair of liftings of X and Y together with a lifting of f . \square

3.7 A few applications of functoriality

We finish our considerations in this chapter with a few applications of functoriality of obstruction classes interesting in the context of mod p^2 and Frobenius liftability.

Lemma 3.7.1. *For any scheme X defined over a perfect field k of characteristic p there exists an obstruction $\sigma_{X/k} \in \text{Ext}^2(L_{X/k}, \mathcal{O}_X)$ whose vanishing is sufficient and necessary for existence of $W_2(k)$ -lifting. The obstruction is functorial, i.e., for any morphism $g : X \rightarrow Y$ the following diagrams is commutative:*

$$\begin{array}{ccc} Lg^*L_{Y/k} & \xrightarrow{Lg^*\sigma_Y} & Lg^*\mathcal{O}_Y[2] \\ \downarrow dg & & \downarrow \simeq \\ L_{X/k} & \xrightarrow{\sigma_X} & \mathcal{O}_X[2], \end{array} \quad \begin{array}{ccc} L_{Y/k} & \xrightarrow{\sigma_Y} & \mathcal{O}_Y[2] \\ \downarrow dg & & \downarrow g^\#[2] \\ Rg_*L_{X/k} & \xrightarrow{Rg_*\sigma_X} & Rg_*\mathcal{O}_X[2]. \end{array}$$

In particular $W_2(k)$ -liftability descends along finite surjective maps of degree prime to the characteristic of k .

Proof. This is a direct consequence of Lemma 3.4.2. The final remark follows from existence of the trace maps splitting $g^\#$. \square

Lemma 3.7.2. *For any scheme X/k together with a $W_2(k)$ -lifting X' there exists an obstruction $\sigma_{X'}^F \in \text{Ext}^1(F_{X/k}^* L_{X^{(1)}/k}, \mathcal{O}_X)$ to lifting of relative Frobenius $F_{X/k} : X \rightarrow X^{(1)}$ to an infinitesimal flat thickening $F'_{X/k} : X' \rightarrow X'^{(1)}$ which satisfies the following functoriality property. For any $g' : X' \rightarrow Y'$ restricting to a given $g : X \rightarrow Y$ the obstructions $\sigma_{X'}^F$ and $\sigma_{Y'}^F$ satisfy:*

$$\begin{array}{ccc} Lg^* F_{Y/k}^* L_{Y^{(1)}/k} & \xrightarrow{Lg^* \sigma_{Y'}^F} & Lg^* \mathcal{O}_Y[1] \\ \downarrow F_{X/k}^* dg^{(1)} & & \downarrow \simeq \\ F_{X/k}^* L_{X^{(1)}/k} & \xrightarrow{\sigma_{X'}^F} & \mathcal{O}_X[1] \end{array} \quad \begin{array}{ccc} F_{Y/k}^* L_{Y^{(1)}/k} & \xrightarrow{\sigma_{Y'}^F} & \mathcal{O}_Y[1] \\ \downarrow d_g^{(1)} & & \downarrow g^\#[1] \\ Rg_* F_{X/k}^* L_{X^{(1)}/k} & \xrightarrow{Rg_* \sigma_{X'}^F} & Rg_* \mathcal{O}_X[1], \end{array}$$

where $d_g^{(1)} : F_{Y/k}^* L_{Y^{(1)}/k} \rightarrow Rg_* F_{X/k}^* L_{X^{(1)}/k}$ is the adjoint of the mapping $F_{X/k}^* dg^{(1)}$. In particular, if g is affine and $g^\# : \mathcal{O}_Y \rightarrow g_* \mathcal{O}_X$ splits then the Frobenius lifting of X descends to Y .

Proof. We apply Lemma 3.4.4 to the case $X_1 = Y$, $Y_1 = Y^{(1)}$, $X_2 = X$, $Y_2 = X^{(1)}$, $f_1 = F_{Y/k}$, $f_2 = F_{X/k}$, and then use the adjunction as in Lemma 3.4.2. The second part follows from the functoriality diagram and the existence of a splitting of $g^\#$. \square

As we suggested in Remark 1.4.3 under the assumption of k being perfect, the vanishing of the obstruction class $\sigma_X^F \in \text{Ext}^1(F_{X/k}^* L_{X^{(1)}/k}, \mathcal{O}_X)$ is sufficient for existence of a lifting of the absolute Frobenius F_X over the Frobenius of the ring of Witt vectors of length two.

As an exemplary application of Lemma 3.7.2 we obtain a corollary:

Corollary 3.7.3. *Let $j : U \rightarrow X$ be the inclusion of an open subset such that complement $Z = X \setminus U$ is of codimension ≥ 3 and X satisfies property S_3 at any point of Z . Then, X admits a Frobenius lifting if and only if U does.*

Proof. Clearly, it suffices to show that Frobenius liftability of U implies liftability for X . For this purpose we observe that by Lemma 3.6.15 the deformation functors of U and X are isomorphic. Hence, any lifting $U' \in \text{Def}_U(W_2(k))$ extends to a lifting $X' \in \text{Def}_X(W_2(k))$ and consequently we may apply functoriality of obstructions Lemma 3.7.2 to the inclusion $U' \rightarrow X'$. We obtain a diagram:

$$\begin{array}{ccc} F_{X/k}^* L_{X^{(1)}/k} & \xrightarrow{\sigma_{X'}^F} & \mathcal{O}_X[1] \\ \downarrow d_j^{(1)} & & \downarrow j^\#[1] \\ Rj_* F_{U/k}^* L_{U^{(1)}/k} & \xrightarrow{Rj_* \sigma_{U'}^F} & Rj_* \mathcal{O}_U[1]. \end{array}$$

By assumption $\sigma_{U'}^F = 0$ and therefore $j^\# \circ \sigma_{X'}^F = 0$. To conclude that $\sigma_{X'}^F = 0$, it suffices to show a natural mapping:

$$\text{Ext}^1(F_{X/k}^* L_{X^{(1)}/k}, \mathcal{O}_X) \longrightarrow \text{Ext}^1(F_{X/k}^* L_{X^{(1)}/k}, Rj_* \mathcal{O}_U)$$

coming from the long exact sequence of $\text{Ext}^\bullet(F_{X/k}^* L_{X^{(1)}/k}, -)$ groups associated to the distinguished triangle

$$\mathcal{O}_X \longrightarrow Rj_* \mathcal{O}_U \longrightarrow R\underline{\Gamma}_Z(X, \mathcal{O}_X)[1] \longrightarrow \mathcal{O}_X[1]$$

is injective. This follows from the vanishing $\text{Ext}^1(F_{X/k}^* L_{X^{(1)}/k}, R\underline{\Gamma}_Z(X, \mathcal{O}_X)[1]) = 0$, which we prove along the lines of the proof of Lemma 3.6.15 (this requires that any point in the complement $X \setminus U$ satisfies the property S_2). \square

We shall now relate the Frobenius liftability of a normal projective scheme and a cone over its sufficiently ample embedding.

Proposition 3.7.4. *Let X be a normal projective scheme defined over a perfect field of characteristic $p > 0$. Then for a sufficiently ample embedding $X \subset \mathbb{P}^n$ the Frobenius liftability of $C = \text{Cone}_{X, \mathbb{P}^n}$ implies the Frobenius liftability of X . Moreover, for any smooth scheme X satisfying $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ Frobenius liftability of X is equivalent to the Frobenius liftability of a cone over a sufficiently ample embedding.*

Proof. The first part we proceed as follows. Let $C = \text{Cone}_{X, \mathbb{P}^n}$ be the cone over an embedding of X such that the functor $\phi : \text{Hilb}_{X, \mathbb{P}^n} \rightarrow \text{Def}_C$ is smooth (cf. Corollary 3.6.18). Moreover, let U be the cone without the vertex \mathfrak{m} . Assume C is Frobenius liftable, i.e., there exists a lifting $\tilde{C} \in \text{Def}_C(W_2(k))$ admitting a lifting of Frobenius. By the smoothness of ϕ we obtain a (potentially non-unique) diagram:

$$\begin{array}{ccc} C & \longrightarrow & \tilde{C} \\ & \nwarrow & \nwarrow \\ & U & \longrightarrow \tilde{U} \\ & \downarrow \pi & \downarrow \tilde{\pi} \\ X & \longrightarrow & \tilde{X} \end{array}$$

By Lemma 3.7.2 we see that the obstruction classes to existence of a Frobenius lifting for X and U fit into a commutative diagram:

$$\begin{array}{ccc} F_{X/k}^* L_{X^{(1)}/k} & \xrightarrow{\sigma_X^F} & \mathcal{O}_X[1] \\ \downarrow d\pi^{(1)} & & \downarrow \pi^\# [1] \\ R\pi_* F_{U/k}^* L_{U^{(1)}/k} & \xrightarrow{R\pi_* \sigma_U^F} & R\pi_* \mathcal{O}_U[1] \end{array} \quad \left. \begin{array}{c} \nearrow \\ \nearrow \end{array} \right)$$

where the right most arrow is a splitting of $\pi^\#$ induced by the \mathbb{G}_m -bundle structure of π . We assumed that \tilde{C} admits a lifting a therefore $\sigma_U^F = 0$. By the existence of splitting $\sigma_X^F = 0$ and therefore X is Frobenius liftable.

For the second statement, we assume that X is Frobenius liftable, i.e., there exists a lifting $\tilde{X} \in \text{Def}_X(W_2(k))$ such that $\sigma_X^F = 0$. By Lemma 3.6.13 and Serre vanishing we obtain an embedding of X into \mathbb{P}^n such that $\psi : \text{Hilb}_{X, \mathbb{P}^n} \rightarrow \text{Def}_X$ is smooth and $H^1(X, \mathcal{O}_X(k)) = H^2(X, \mathcal{O}_X(k)) = 0$ for any $k \neq 0$. By smoothness of ψ we see that \tilde{X} arises as an embedded deformation and therefore induces a compatible deformation $\tilde{C} \in \text{Def}_C(W_2(k))$ of the cone C . By Corollary 3.7.3, in order to prove that \tilde{C} admits

a Frobenius lift, it suffices to infer it for \tilde{U} . By functoriality of obstructions to lifting Frobenius we obtain a diagram:

$$\begin{array}{ccc}
F_{U/k}^* L\pi^{(1)*} L_{X^{(1)}/k} & \simeq L\pi^* F_{X/k}^* L_{X^{(1)}/k} & \xrightarrow{L\pi^* \sigma_X^F} L\pi^* \mathcal{O}_X[1] \\
\downarrow F_{U/k}^* d\pi^{(1)} & & \downarrow \simeq \\
F_{U/k}^* L_{U^{(1)}/k} & \xrightarrow{\sigma_U^F} & \mathcal{O}_U[1].
\end{array}$$

We consider a Frobenius pullback of a distinguished triangle of cotangent complexes associated to the \mathbb{G}_m -bundle $\pi^{(1)} : U^{(1)} \rightarrow X^{(1)}$:

$$F_{U/k}^* L\pi^{(1)*} L_{X^{(1)}/k} \longrightarrow F_{U/k}^* L_{U^{(1)}/k} \longrightarrow F_{U/k}^* L_{U^{(1)}/X^{(1)}} \simeq \mathcal{O}_U \longrightarrow F_{U/k}^* L\pi^{(1)*} L_{X^{(1)}/k}[1],$$

where the isomorphism in the middle follows from the standard property of Zariski locally trivial \mathbb{G}_m -bundles. By the long exact sequence of $\text{Ext}(-, \mathcal{O}_U)$ groups we obtain an exact sequence:

$$\dots \longrightarrow \text{Ext}^1(\mathcal{O}_U, \mathcal{O}_U) \longrightarrow \text{Ext}(F_{U/k}^* L_{U^{(1)}/k}, \mathcal{O}_U) \longrightarrow \text{Ext}^1(F_{U/k}^* L\pi^{(1)*} L_{X^{(1)}/k}, \mathcal{O}_U) \longrightarrow \dots,$$

and therefore it suffices to prove that $\text{Ext}^1(\mathcal{O}_U, \mathcal{O}_U) = 0$. This follows from the isomorphisms $\text{Ext}^1(\mathcal{O}_U, \mathcal{O}_U) \simeq H^1(U, \mathcal{O}_U) \simeq H^1(X, \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_X(k))$ and the assumptions. \square

Chapter 4

Basics of characteristic p geometry

Before proceeding to the considerations concerning deformation theory, we present a few tools useful in the study of characteristic p schemes. In §4.1 and §4.2, we give a very brief introduction to p -adic Weil cohomology theories and the notion of ordinarity. Then, in §4.3 we recall the so-called Cartier isomorphism describing cohomology sheaves of the de Rham complex. In §4.4, following the exposition of [MR85], we review the definition and basic properties of Frobenius split schemes. Finally, in §4.5, we focus on singularities of characteristic $p > 0$ schemes. We recall the standard definitions of so-called F -singularities and present a few useful examples.

4.1 Characteristic p cohomology theories

Since we deal with pathological behavior of characteristic p schemes, we need the following tools, which allows one to control their interesting cohomological behaviour. We begin with an overview of Weil cohomology theories. Let k be a base field, and let F be a characteristic zero field, considered as ring of coefficients for the cohomology theory.

Definition 4.1.1. A *Weil cohomology theory* is a contravariant functor H^\bullet from the category **Smooth** $_k$ of smooth varieties to the category **GrAlg** $_F$ of graded commutative algebras over F together with the following structure:

1. for every X , a linear trace map $Tr_{H^\bullet X}: H^{2\dim X}(X) \rightarrow F$
2. for every X , and for every closed irreducible subvariety $Z \rightarrow X$ of codimension c , a cohomology class $cl(Z) \in H^{2c}(X)$

satisfying certain conditions concerning pullback maps f^* , for every $f: X \rightarrow Y$, and multiplication $\cup: H^\bullet(X) \otimes_F H^\bullet(X) \rightarrow H^\bullet(X)$ (see [And04, Section 3.3] for the details).

The most elementary example of a Weil cohomology theory is the \mathbb{C} -linear singular cohomology theory $H_{\text{sing}}(X, \mathbb{C})$ defined varieties over \mathbb{C} . Unfortunately, due to its inherently transcendental nature and relative coarseness of the Zariski topology, it cannot be generalized to arithmetic or characteristic p geometry.

4.1.1 ℓ -adic cohomology

A way to overcome this problem, proposed by Grothendieck, is to generalize the concepts of an open covering in order to obtain some additional flexibility. This leads to the

notions of a site, a topos (category of sheaves on a site) and cohomology on a topos. In the special case, when the site is generated by jointly surjective étale morphisms, we obtain the so-called étale cohomology theory $H_{\text{ét}}^{\bullet}(X, -)$ which behaves well for torsion coefficients. Fixing a prime ℓ , and considering a limit

$$H_{\text{ét}}^{\bullet}(X, \mathbb{Q}_{\ell}) \stackrel{\text{def}}{=} \left(\lim_n H_{\text{ét}}^{\bullet}(X_{\bar{k}}, \mathbb{Z}/\ell^n \mathbb{Z}) \right) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell},$$

we obtain the so-called ℓ -adic cohomology theory with values in \mathbb{Q}_{ℓ} . This theory works well only if ℓ is relatively prime to the characteristic of the base field k , and in this case gives a Weil cohomology theory (see [Mil80]). Moreover in characteristic zero, for torsion coefficients, it is related with the singular cohomology by means of the comparison isomorphism of Artin.

Example 4.1.2. Let k be a field of characteristic p . By induction, using the Artin-Schreier sequence $0 \rightarrow \mathbb{F}_p \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$, we see that $H^i(\mathbb{P}_k^1, \mathbb{Z}/p^n) = 0$, for $i > 0$ and therefore $H_{\text{ét}}^i(X, \mathbb{Q}_p) = 0$.

4.1.2 Crystalline cohomology

Let us consider the category of smooth projective schemes defined over a perfect field of characteristic p . In order to remedy the problem described above and capture the *p-adic* behaviour of cohomology, based on ideas of Grothendieck, Berthelot (see [Ber74]) constructs the so-called crystalline cohomology $H_{\text{crys}}^{\bullet}(X/W)$ with values in the ring of Witt vectors $W(k)$. As above, it arises as a limit of cohomology groups over torsion rings. More precisely, to every scheme one associates a sequence of ringed topoi $(X_{\text{crys}, n}, \mathcal{O}_{X_{\text{crys}, n}/W_n})$, which lead to a sequence of cohomology groups $H_{\text{crys}}^{\bullet}(X/W_n) \stackrel{\text{def}}{=} H^{\bullet}(X_{\text{crys}, n}, \mathcal{O}_{X_{\text{crys}, n}/W_n})$. Taking the limit as $n \rightarrow \infty$, one define

$$H_{\text{crys}}^{\bullet}(X/W) \stackrel{\text{def}}{=} \lim_n H_{\text{crys}}^{\bullet}(X/W_n).$$

It turns out that up to p -torsion the functor $H_{\text{crys}}^{\bullet}(X/W)$ gives a Weil cohomology theory with values in the fraction field $\text{Frac } W(k)$. Moreover, by means of the short exact sequence:

$$0 \rightarrow H_{\text{crys}}^n(X/W) \otimes_W k \rightarrow H_{\text{dR}}^n(X/k) \rightarrow \text{Tor}_W^1(H_{\text{crys}}^{n+1}(X/W), k) \rightarrow 0,$$

its torsion describes interesting phenomena of characteristic p geometry (e.g., lack of Hodge degeneration [Mum61]). In the special case, when a variety X admits a lifting \mathcal{X} over the ring of Witt vectors $W(k)$, the crystalline cohomology is isomorphic to the relative de Rham cohomology $H_{\text{dR}}^{\bullet}(\mathcal{X}/W)$. A good reference for the above is the book [BO78].

4.1.3 de Rham–Witt complex and Hodge–Witt cohomology

In the interesting cases, the assumption of existence of a Witt vector lifting is usually not satisfied. Furthermore the crystalline site is difficult to grasp, and therefore it is natural to ask whether crystalline cohomology can be computed from coherent cohomology on a scheme closely related to X . Affirmative answer to the question is provided by the construction of the so-called Rham–Witt procomplex $W_n \Omega_X^{\bullet}$. The relation with the crystalline topos is explained by the quasi-isomorphism:

$$W_n \Omega_X^{\bullet} \simeq Ru_{n*} \mathcal{O}_{X_{\text{crys}, n}/W_n},$$

where $u_n: X_{\text{crys},n} \rightarrow X$ is a natural morphism of topoi. Consequently, the hypercohomology groups $H^\bullet(X, W_n \Omega_X^\bullet)$ compute the crystalline cohomology $H_{\text{crys}}^\bullet(X/W_n)$. In the limit ([Ill79, Proposition 2.1, p. 607]), one obtains a complex of W -modules $W\Omega_X^\bullet$ and isomorphisms:

$$H_{\text{crys}}^\bullet(X/W) \simeq \lim_n H_{\text{crys}}^\bullet(X/W_n) \simeq \lim_n H^\bullet(X, W_n \Omega_X^\bullet) \simeq H^\bullet(X, W\Omega_X^\bullet).$$

The natural filtration $F^i W\Omega_X^\bullet = W\Omega_X^{\geq i}$ of the de Rham–Witt complex leads to a spectral sequence

$$H^j(X, W\Omega_X^i) \Rightarrow H_{\text{crys}}^n(X/W),$$

which degenerates after twisting with the fraction field of W . In order to control the behaviour of the p -torsion one defines the Hodge–Witt cohomology

$$H_{\text{HW}}^n(X) = \bigoplus_{i+j=n} H^j(X, W\Omega_X^i).$$

Generalized from the ring of Witt vectors, the de Rham–Witt complex, and hence the Hodge–Witt cohomology groups, admits a pair of operators F, V (Frobenius and Verschiebung) satisfying the relationship $FV = p = VF$. All the above results can be found in [Ill79].

4.1.4 Blow-up formulas

In this section, we review formulas for the cohomology groups of the blow-up of a smooth proper scheme X along a smooth subscheme Z . It turns out that to deduce the blow-up formulas for different cohomology theories it suffices to give a single *motivic decomposition* in the category of Chow motives \mathbf{CM}_k (see [And04, Section 4]). More precisely, there exists a functor $M: \mathbf{Smooth}_k \rightarrow \mathbf{CM}_k$ such that every Weil cohomology theory (in fact every cohomology theory admitting cycle class maps and actions by correspondences) factors through M .

Proposition 4.1.3 ([Voe00, 3.5.3]). *Suppose that X is a smooth proper scheme over a field k , $Z \subset X$ a smooth closed subscheme of codimension c . Then there is a decomposition of Chow motives:*

$$M(\text{Bl}_Z X) = M(X) \bigoplus_{i=1}^{c-1} M(Z)(i)[2i].$$

As an immediate corollary we see that

Corollary 4.1.4. *Suppose that X is a smooth proper scheme over a field k , $Z \subseteq X$ a smooth closed subscheme of codimension c . Let H^n denote one of the following families of functors of smooth projective varieties X :*

- (1) $H_{\text{ét}}^n(X \otimes \bar{k}, \mathbb{Z}_\ell)$ for some ℓ invertible in k , treated as a $\text{Gal}(\bar{k}/k)$ -module,
- (2) (if k is perfect of characteristic $p > 0$) $H_{\text{crys}}^n(X/W(k))$, the integral crystalline cohomology, a $W(k)$ -module with a σ -linear endomorphism induced by the Frobenius,
- (3) $H_{\text{dR}}^n(X) = H^n(X, \Omega_{X/k}^\bullet)$, the algebraic de Rham cohomology, endowed with the Hodge filtration,

- (4) $H_{Hdg}^n(X) = \bigoplus_{p+q=n} H^q(X, \Omega_{X/k}^p)$, Hodge cohomology, a graded k -vector space,
- (5) $H_{HW}^n(X) = \bigoplus_{p+q=n} H^q(X, W\Omega_X^p)$, Hodge–Witt cohomology, a graded $W(k)$ -module endowed with $\sigma^{\pm 1}$ -linear endomorphisms F and V satisfying $FV = p = VF$,
- (6) (if n is even) $A^{n/2}(X)$, the $\frac{n}{2}$ -th Chow group of X (treated as an abelian group).

Moreover, let $-(n)$ denote the Tate twist, i.e., the tensor product with $H^2(\mathbb{P}^1)^{\otimes n}$ in the appropriate tensor category. Then there is a natural isomorphism of objects in the appropriate category as listed above

$$H^n(\mathrm{Bl}_Z(X)) = H^n(X) \oplus \bigoplus_{i=1}^{c-1} H^{n-2i}(Z)(i).$$

Proof. This follows from Proposition 4.1.3 and the fact that the cohomology theories H above all admit cycle class maps and actions by correspondences. For ℓ -adic and crystalline cohomology this is well-known, and for Hodge and Hodge–Witt cohomology it follows from the work of Chatzistamatiou and R ulling [CR11]. \square

4.2 Hodge–de Rham spectral sequence, ordinarity and the Hodge–Witt property

Let X be a smooth proper scheme over k . The first hypercohomology spectral sequence of the de Rham complex $\Omega_{X/k}^\bullet$,

$$E_1^{ij} = H^j(X, \Omega_{X/k}^i) \Rightarrow H_{dR}^{i+j}(X/k) := H^{i+j}(X, \Omega_{X/k}^\bullet),$$

is called the *Hodge–de Rham spectral sequence* of X . We say that it *degenerates* if it degenerates on the first page, i.e., there are no nonzero differentials. As X is proper, the cohomology groups are finite dimensional, and hence the degeneration is equivalent to the condition that

$$\dim H_{dR}^n(X/k) = \sum_{p+q=n} \dim H^q(X, \Omega_{X/k}^p) \quad \text{for all } n \geq 0. \quad (4.1)$$

The Hodge–de Rham spectral sequence of X degenerates if k is of characteristic zero, or if $\dim X \leq p = \mathrm{char} k$ and X lifts to $W_2(k)$ (cf. Corollary 5.2.3).

Definition 4.2.1. The scheme X is called *ordinary* (in the sense of Bloch and Kato) if the Frobenius $F: H^q(X, W\Omega_X^p) \rightarrow H^q(X, W\Omega_X^p)$ on Hodge–Witt cohomology is bijective for all p and q

Let us recall that B_X^i denote the sheaves of coboundaries in the de Rham complex $\Omega_{X/k}^\bullet$.

Proposition 4.2.2 ([BK86, Proposition 7.3]). *A scheme X is ordinary if and only if $H^j(X, B_X^i) = 0$*

Remark 4.2.3. In fact in [BK86, Proposition 7.3]) one can find a few other equivalent characterizations of ordinarity.

Definition 4.2.4 (Hodge–Witt). We say that X is Hodge–Witt if for $i, j \geq 0$, the Hodge–Witt cohomology groups $H^i(X, W\Omega_X^j)$ are finitely generated $W(k)$ -modules. It is called *Hodge–Witt* if the Hodge–Witt groups $H^q(X, W\Omega_X^p)$ are finitely generated $W(k)$ -modules.

It follows from [IR83, IV 4] that X is Hodge–Witt if it is ordinary, and that $X \times Y$ is ordinary if X and Y are.

Corollary 4.2.5. *Suppose that X is a smooth proper scheme over a field k , $Z \subseteq X$ a smooth closed subscheme of codimension > 1 . Then*

1. *The Hodge–de Rham spectral sequences of Z and X degenerate if and only if the Hodge–de Rham sequence of $\mathrm{Bl}_Z X$ degenerates.*
2. *The scheme $\mathrm{Bl}_Z(X)$ is ordinary (resp. Hodge–Witt) if and only if both X and Y are ordinary (resp. Hodge–Witt).*

Proof. The first assertion follows from Corollary 4.1.4 for H_{dR}^n and H_{Hdg}^n and (4.1). For the latter, use Corollary 4.1.4 for H_{HW}^n and the characterizations given above. \square

Definition 4.2.6 (Weakly ordinary). Let X be a smooth scheme over a perfect field k . We say that X is *weakly ordinary* if the natural Frobenius action $F^*: H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$ is bijective, for every $i \geq 0$.

Proposition 4.2.7. *Let X be an abelian variety. Then the following are equivalent:*

- (1) *X is ordinary,*
- (2) *X is weakly ordinary,*
- (3) *the natural Frobenius action $F^*: H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X)$ is bijective,*
- (4) *the natural Frobenius action $F^*: H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X)$ is bijective.*

Proof. The equivalence of (1) and (2) is the content of [MS87, Lemma 1.1]. For the equivalence of (2), (3) and (4) we first observe that for an abelian variety $F^*: H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$ can be identified, using Künneth formula, with the i -th exterior power of $F^*: H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X)$. Therefore, $F^*: H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$ is bijective if and only if $F^*: H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X)$ is bijective if and only if $F^*: H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X)$ is bijective. \square

Proposition 4.2.8. *X is weakly ordinary if and only if $H^i(X, B_X^1) = 0$ for any $i \geq 0$.*

Proof. This is a straightforward consequence of the long exact sequence of cohomology for

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow B_X^1 \longrightarrow 0. \quad \square$$

4.3 Cartier isomorphism

We begin with a classical result of Cartier describing cohomology sheaves of de Rham complex $\Omega_{X/S}^\bullet$ of a given smooth morphism $X \rightarrow S$ of schemes defined over k . We denote by $Z_{X/S}^i$ (resp. $B_{X/S}^i$) the sheaves of cocycles (resp. coboundaries) in the $\mathcal{O}_{X(1)}$ -linear de Rham complex $F_{X/S*}\Omega_{X/S}^\bullet$. For the sake of brevity, after [MR85, Section 2], we adopt the following convention. Let x_1, \dots, x_n be a system of parameters around a given closed point x of a smooth scheme X over k , and let $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a sequence of rational numbers. By $\underline{x}^{\underline{\alpha}}$ we mean the element $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ iff all α_i are integral and 0 otherwise. Moreover, we denote by $d\underline{x}$ the differential form $dx_1 \wedge \cdots \wedge dx_n$ and by $1/d\underline{x}$ the corresponding local section of ω_X^{-1} .

Theorem 4.3.1. *Let $X \rightarrow S$ be a smooth morphism of schemes defined over a perfect field k of characteristic $p > 0$. Then the association $\mathcal{O}_{X^{(1)}} \ni f \otimes s \mapsto sf^{p-1}df \pmod{B_{X/S}^1} \in \mathcal{H}^1(F_{X/S*}\Omega_{X/S}^\bullet)$ is a k -derivation such that the corresponding $\mathcal{O}_{X^{(1)}}$ -linear mapping given by $d(f \otimes s) \mapsto sf^{p-1}df$ (traditionally called inverse Cartier transformation and denoted by C^{-1})*

$$C^{-1}: \Omega_{X^{(1)}/S}^1 \rightarrow \mathcal{H}^1(F_{X/S*}\Omega_{X/S}^\bullet) = Z_{X/S}^1/B_{X/S}^1$$

is an isomorphism. Moreover, the natural graded commutative extension

$$C^{-1}: \bigoplus \Omega_{X^{(1)}/S}^i[-i] \rightarrow \bigoplus \mathcal{H}^i(F_{X/S*}\Omega_{X/S}^\bullet)[-i]$$

defined by the formula

$$d(f_1 \otimes 1) \wedge \dots \wedge d(f_i \otimes 1) \mapsto C^{-1}(df_1) \wedge \dots \wedge C^{-1}(df_i) \pmod{B_{X/S}^i}$$

is an isomorphism of sheaves of graded commutative algebras.

Proof. For the proof, see [Kat70, Theorem 7.2] □

As a consequence, we obtain a system of short exact sequences of $\mathcal{O}_{X^{(1)}}$ -modules

$$0 \longrightarrow B_{X/S}^n \longrightarrow Z_{X/S}^n \longrightarrow \Omega_{X^{(1)}/S}^n \longrightarrow 0.$$

In particular, for $n = \dim X$ we get a surjective morphism $F_*\omega_{X^{(1)}/S} = Z_{X/S}^{\dim X} \rightarrow \Omega_{X^{(1)}/S}^{\dim X} = \omega_{X^{(1)}/S}$ which is traditionally called trace of Frobenius and denoted by $\mathrm{Tr}_{X/S}$.

Remark 4.3.2. For $S = \mathrm{Spec}(k)$, where k is a perfect field, one can identify the schemes X and $X^{(1)}$ via the natural k -semilinear isomorphism $W_{X/k}$ (cf. §1.4) and interpret the above theorem as a statement about sheaves on a given scheme X . In particular, in this manner we obtain inverse Cartier transformations $\Omega_{X/k}^i \rightarrow Z_X^i/B_X^i$, where Z_X^i (resp. B_X^i) is an \mathcal{O}_X -subsheaf of $F_*\Omega_{X/k}^i$ of closed (resp. exact) differential i forms, and a system of short exact sequences of \mathcal{O}_X -modules

$$0 \longrightarrow B_X^i \longrightarrow Z_X^i \longrightarrow \Omega_{X/k}^i \longrightarrow 0.$$

Again, in the case $i = \dim X$ we get a surjective morphism $F_*\omega_X = Z_X^{\dim X} \rightarrow \Omega_{X/k}^{\dim X} = \omega_X$ referred to as THE trace of Frobenius and denoted by Tr_X .

It turns out that the trace of Frobenius can be locally described as follows.

Proposition 4.3.3. *Let x be a closed point of X . Then locally around x there exists a coordinate system x_1, \dots, x_n such that*

$$\mathrm{Tr}_X(\underline{x}^\alpha d\underline{x}) = \underline{x}^{(\alpha-p+1)/p} d\underline{x}.$$

Proof. First, we observe that the inverse Cartier transformation is defined by the formula $d\underline{x} \mapsto \underline{x}^{p-1}d\underline{x}$. Moreover, by a direct computation we see that for any $\underline{\alpha}$ such that $(\alpha - p + 1)/p$ is not integral the differential form $\underline{x}^\alpha d\underline{x}$ is exact. Combining these two observations, we obtain the claim. □

4.4 Frobenius splittings

□ We now recall a few basic facts concerning the notion of Frobenius splitting. The notion first appeared in [MR85] in the study of properties of homogeneous spaces and Schubert varieties.

Definition 4.4.1 (Frobenius-split). We say that a characteristic $p > 0$ scheme X is *Frobenius-split* if the natural mapping $F^\#: \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ is split, i.e., there exists an \mathcal{O}_X -linear mapping $s: F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ such that $s \circ F^\#$ equals the identity on \mathcal{O}_X .

4.4.1 Consequence of existence of a Frobenius splitting

Here, we prove a few properties of Frobenius-split schemes.

Proposition 4.4.2 ([BK05, Proposition 1.2.1]). *Any Frobenius-split scheme X is reduced.*

Proof. Since X is Frobenius-split, there exists an \mathcal{O}_X -linear mapping $s: F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ such that $s(g^p) = g$ for any open $U \subset X$ and $g \in \mathcal{O}_X(U)$. To prove our claim, it suffices to prove that any nilpotent section $f \in \mathcal{O}_X(U)$ is 0. Potentially taking a higher multiple we assume that $f^{p^\nu} = 0$ and $\nu \geq 0$ is the smallest possible. If $\nu \geq 1$ then $f^{p^{\nu-1}} = s((f^{p^{\nu-1}})^p) = s(f^{p^\nu}) = 0$ contradicting the choice of ν . Therefore $\nu = 0$ and we are done. □

Now, we present a standard characteristic p technique useful in proving vanishing theorems.

Proposition 4.4.3 ([BK05, Lemma 1.2.7]). *Let X be a smooth Frobenius-split scheme over k and let \mathcal{L} be an ample line bundle. Then for any $i > 0$ we have $H^i(X, \omega_X \otimes \mathcal{L}) = 0$.*

Proof. First, using Serre duality, we observe that it suffices to prove that $H^j(X, \mathcal{L}^{-1})$ for any $j < \dim X$. Since $F^\#: \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ is split, we obtain an injective mapping

$$H^j(X, \mathcal{L}^{-1}) \rightarrow H^j(X, F_*\mathcal{O}_X \otimes \mathcal{L}^{-1}) \simeq H^j(X, F_*\mathcal{L}^{-p}) = H^j(X, \mathcal{L}^{-p}),$$

where the isomorphism follows from the projection formula. By iterating the above procedure, we get an injection $H^j(X, \mathcal{L}^{-1}) \rightarrow H^j(X, \mathcal{L}^{-p^\nu})$ for any $\nu \geq 1$. Since \mathcal{L} is ample, by Serre vanishing, we see that $H^j(X, \mathcal{L}^{-p^\nu}) = H^{\dim X - j}(X, \omega_X \otimes \mathcal{L}^{p^\nu}) = 0$ for ν large enough. Hence, the proof is finished. □

4.4.2 Criteria for splittings

Following the presentation in [MR85, Section 2], we now present a few criteria for existence of a Frobenius splitting. Throughout X denotes a smooth variety defined over k . First, we sketch the proof of so-called Grothendieck's duality in the case of Frobenius morphism. Let \mathcal{M} be a locally free sheaf on X . The \mathcal{O}_X -linear association

$$F_*\mathcal{M} \otimes F_*\mathcal{H}om(\mathcal{M}, \omega_X) \ni m \otimes \phi \mapsto \text{Tr}_X(\phi(m)) \in \omega_X$$

gives rise to a mapping $d_{\mathcal{M}}: F_*\mathcal{H}om(\mathcal{M}, \omega_X) \rightarrow \mathcal{H}om(F_*\mathcal{M}, \omega_X)$ defined by the formula $\phi \mapsto \text{Tr}_X \circ \phi$. It turns out that $d_{\mathcal{M}}$ is an isomorphism.

Proposition 4.4.4. *Let \mathcal{M} be a locally free sheaf on X . Then the natural map*

$$d_{\mathcal{M}}: F_* \mathcal{H}om(\mathcal{M}, \omega_X) \rightarrow \mathcal{H}om(F_* \mathcal{M}, \omega_X)$$

is an isomorphism.

Proof. The statement is local on X and therefore it suffices to prove it on an open $U \subset X$ trivializing \mathcal{M} such that there exists a system of local parameters x_1, \dots, x_n . The sheaf $F_* \mathcal{H}om(\mathcal{M}, \omega_X)$ is then locally generated by homomorphisms $h_{\underline{\alpha}}: F_{\mathcal{M}} \rightarrow \omega_X$, for $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ satisfying $0 \leq \alpha_i \leq p-1$, defined by the formula $h_{\underline{\alpha}}(\underline{x}^{\underline{\beta}}) = d\underline{x}$ if $\underline{\alpha} = \underline{\beta}$ and 0 otherwise. Using Prop. 4.3.3 we see that $d_{\mathcal{M}}(h_{\underline{\alpha}})(\underline{x}^{\underline{\gamma}}) = \underline{x}^{(\alpha+\gamma-p+1)/p} d\underline{x}$ and therefore $d_{\mathcal{M}}(h_{\underline{\alpha}}) = g_{\underline{p-1-\alpha}}$, where $g_{\underline{\gamma}}$ is the generating set for $\mathcal{H}om(F_* \mathcal{M}, \omega_X)$ defined by the formula $g_{\underline{\gamma}}(\underline{x}^{\underline{\beta}}) = d\underline{x}$ if $\underline{\gamma} = \underline{\beta}$ and 0 otherwise. \square

As a corollary we see that

Corollary 4.4.5. *There exists a natural isomorphism $F_* \omega_X^{1-p} \simeq \mathcal{H}om(F_* \mathcal{O}_X, \mathcal{O}_X)$ described by the association $\underline{x}^{\underline{\alpha}}/d\underline{x}^{p-1} \mapsto (\underline{x}^{\underline{\beta}} \mapsto \underline{x}^{(\alpha+\beta-p+1)/p})$. The section $\eta \in F_* \omega_X^{1-p}$ gives a splitting if and only if there exists a point x and a system of local parameters x_1, \dots, x_n such that \underline{x}^{p-1} appears with non-zero coefficient in the expansion of $\eta_x \in \omega_{X,x} \otimes \widehat{\mathcal{O}_{X,x}}$, where $\widehat{\mathcal{O}_{X,x}}$ is the completion of the local ring of x .*

Proof. This is a direct consequence of the above Prop. 4.4.4. For a complete proof see [MR85, Prop. 5 and Prop. 6]. \square

Using Serre duality one can prove the following simple criterion.

Proposition 4.4.6. *Let X be a smooth scheme over k . Then X is Frobenius-split if and only if the natural mapping $H^{\dim X}(X, \omega_X) \rightarrow H^{\dim X}(X, \omega_X^{\otimes p})$ induced by the $\mathcal{O}_X \otimes \omega_X \rightarrow F_* \mathcal{O}_X \otimes \omega_X$ is injective (or equivalently nonzero).*

Proof. For the proof, see [MR85, Proposition 9]. \square

4.4.3 Canonical splittings for abelian and Calabi-Yau varieties

We shall now apply the above criteria in case of weakly ordinary (cf. Definition. 4.2.6) Calabi-Yau manifolds to obtain canonical Frobenius splittings of such varieties. We begin with a definition.

Definition 4.4.7 (Calabi-Yau variety). We say that a smooth variety X over a field k is *Calabi-Yau* if $\omega_{X/k} \simeq \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$.

Proposition 4.4.8. *Let X be either a Calabi-Yau or an abelian variety defined over a perfect field k . The X is Frobenius-split if and only if it is weakly ordinary. If a splitting exists then it is unique.*

Proof. First, since X is Calabi-Yau, we see that $H^i(X, \mathcal{O}_X) = 0$ unless $i = 0$ or $i = \dim X$. Therefore X is weakly ordinary if and only if the p -linear action of Frobenius on the top cohomology groups $H^{\dim X}(X, \mathcal{O}_X) \rightarrow H^{\dim X}(X, \mathcal{O}_X)$ is bijective. Using Proposition. 4.4.6 this is equivalent to existence of a Frobenius splitting. To establish uniqueness, we observe that the set of splittings is a torsor under $H^0(X, (B_X^1)^{\vee})$. Using Serre duality the last group is isomorphic to $H^n(X, B_X^1)$ which is zero by weak ordinarity (cf. Proposition. 4.2.8) \square

4.5 F-singularities

In this section, we review the definitions and basic properties of F -singularities, that is, singularity types defined in terms of the Frobenius morphism. In particular, we introduce F -pure, strongly F -regular and F -rational singularities and provide a few illustrating examples. Moreover, we present Fedder's criterion for F -purity and its generalization for strong F -regularity, proven by Glassbrenner. We do not intend to give a thorough exposition of the topic. A reader willing to grasp the general theory can consult the expository works like [TW15, ZKT14].

4.5.1 Convention and basics results

Throughout this section R denotes a noetherian ring of characteristic p . We say that R is F -finite if F_*R is a finitely generated R -module. For example, the F -finiteness assumption is satisfied in the case of essential finite type algebras over a field k of characteristic p . For a reduced F -finite ring R , and for any $e > 0$, we identify $F_*^e R$ with the ring R^{1/p^e} of p^e -th roots of elements of R . In this setting, the Frobenius morphism $r \mapsto r^{p^e}$ is identified with the natural inclusion $r \mapsto r = (r^{1/p^e})^{p^e}$. For any ideal I we denote by $I^{[p^e]}$ the extension of I along the Frobenius morphism, i.e., the ideal generated by p -th powers of generators of I . Using the convention involving p^e -th roots this is an ideal $I \cdot R^{1/p^e}$. We denote by R° the set of elements in R which do not lie in any minimal prime ideal. In what follows we do not distinguish between affine schemes and their associated rings.

The interest in the Frobenius morphism in the context of properties of singular schemes is motivated by the following result of Kunz.

Theorem 4.5.1 ([Kun69]). *Let R be a noetherian ring of characteristic p . Then the Frobenius morphism $F: R \rightarrow R$ is flat (i.e., F_*R is a flat R -module) if and only if R is regular.*

Corollary 4.5.2. *Let X be a locally noetherian scheme over \mathbb{F}_p . Then X is regular if and only if $F_*\mathcal{O}_X$ is a flat \mathcal{O}_X -module.*

Proof. Apply Theorem 4.5.1 to the stalks of \mathcal{O}_X . □

Remark 4.5.3. In order to prove flatness of $F: R \rightarrow R$ it suffices to establish it for every localization in prime or maximal ideals.

We now proceed to the description of four basic characteristic $p > 0$ singularity types: F -purity, F -regularity, F -rationality and F -injectivity.

4.5.2 F -purity

In the course of the proof that ring of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay (cf. [HR76]), Hochster and Roberts introduced the following notion of F -purity.

Definition 4.5.4. We say that a noetherian ring R is F -pure if the Frobenius morphism $R \rightarrow F_*R$ is split, i.e., the scheme $\text{Spec}(R)$ is Frobenius-split (cf. §4.4).

Proposition 4.5.5. *Let R be an F -finite noetherian ring. Then R is F -pure if and only if $R_{\mathfrak{p}}$ (resp. $R_{\mathfrak{m}}$) is F -pure for every prime (resp. maximal) ideal \mathfrak{p} (resp. \mathfrak{m}).*

Proof. The ring R is F -pure if and only if the natural evaluation at $1 \in F_*R$ map

$$\text{ev}_1: \text{Hom}_R(F_*R, R) \ni \phi \mapsto \phi(1) \in R$$

is surjective. Since F_*R is a finitely generated R -module, taking modules of homomorphism $\text{Hom}_R(F_*R, R)$ commutes with localization, and hence we are done for an R -module homomorphism is surjective if and only if its localizations are. \square

The property of F -purity turns out to be weaker than regularity (flatness of the Frobenius by the above Theorem 4.5.1).

Proposition 4.5.6. *Let R be a regular ring then R is F -pure.*

Proof. First, we may assume that R is local with maximal ideal \mathfrak{m} . Over a local ring every finitely generated module is in fact free, and therefore F_*R is free, spanned by the lifts of the tangent space $F_*R/\mathfrak{m}F_*R$. The element $1 \in F_*R$ does not belong to $\mathfrak{m}F_*R$ and therefore we are done. \square

Along the lines of Prop. 4.4.2, we may prove that every F -pure ring is reduced. In [Fed83], Fedder proved the following computational criterion for F -purity of quotients of regular ring.

Lemma 4.5.7 ([Fed83, Proposition 1.7]). *Let S be a F -finite regular ring such that F_*S is a free S -module and let $R = S/I$. Then R is F -pure at the maximal ideal $\mathfrak{m} \supset I$ (i.e. $R_{\mathfrak{m}}$ is F -pure) if and only if $I^{[p]} : I \not\subset \mathfrak{m}^{[p]}$.*

Example 4.5.8. Let R be a noetherian domain, and let $I = (f)$ be a principal ideal. Then $I^{[p]} : I = (f^{p-1})$ and hence $R/(f)$ is F -pure at \mathfrak{m} if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$. As a consequence we see that

- 1) $k[x, y, z]/(x^2 + y^2 + z^{n+1})$ is F -pure for $p \geq 3$,
- 2) $k[x_1, \dots, x_n]/(x_1^2 + \dots + x_n^2)$ is F -pure for every $n \geq 2$ and $p \geq 3$,
- 3) $k[x_1, \dots, x_n]/(x_1^n + \dots + x_n^n)$ is F -pure if and only if $p \equiv 1 \pmod{n}$,
- 4) $k[x_1, \dots, x_n]/(x_1^k + \dots + x_n^k)$ is not F -pure if $k \lceil \frac{p-1}{n} \rceil \geq p$.

We give an exemplary computation in the first case. We suppose, towards the contradiction, that $(x^2 + y^2 + z^{n+1})^{p-1} \in (x^p, y^p, z^p)$. By considering the natural homomorphism $k[x, y, z] \rightarrow k[x, y, z]/(z) \simeq k[x, y]$, we see that $(x^2 + y^2)^{p-1} \in (x^p, y^p)$. But $(x^2 + y^2)^{p-1} = \sum_{i+j=p-1} x^{2i} y^{2j} = x^{p-1} y^{p-1} \not\equiv 0 \pmod{x^p, y^p}$ which is a contradiction.

Proposition 4.5.9 ([SS10, Proposition 5.3]). *Let X be a smooth projective scheme and let \mathcal{L} be an ample line bundle. Then $\text{Cone}_{X, \mathcal{L}}$ is F -pure at the vertex (and hence at every point) if and only if X is Frobenius-split.*

Remark 4.5.10. An F -pure rings is not necessarily Cohen-Macaulay. Indeed, using Prop. 4.5.9 we see that for an abelian variety A over a perfect field k and an ample line bundle \mathcal{L} the cone $\text{Cone}_{A, \mathcal{L}}$ is F -pure if and only if A is Frobenius-split. By Prop. 4.4.6 we see that A is Frobenius-split if and only if the Frobenius action $F^*: H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X)$ is injective. This in turn is equivalent to ordinarity by Prop. 4.2.7. Therefore, $\text{Cone}_{A, \mathcal{L}}$ is F -pure if and only if A is ordinary. On the other hand, using Prop. 2.5.2 we see that $\text{Cone}_{A, \mathcal{L}}$ is not Cohen-Macaulay unless $\dim A = 1$.

4.5.3 F -regularity

In the paper [HH89], Hochster and Huneke distinguished the following class of rings containing F -pure rings.

Definition 4.5.11 (strong F -regularity). Let R be a reduced F -finite ring of characteristic $p > 0$. We say that R is F -regular if for any $c \in R^\circ$ the mapping $R \rightarrow F_*^e R$ defined by the formula $1 \mapsto c$ splits for some $e \gg 0$.

Proposition 4.5.12 ([HH89, 3.1. Theorem]). *Let R be a regular ring. Then R is strongly F -regular.*

Therefore the class of strongly F -regular rings refines the inclusion of regular rings into F -pure rings. In the case of quotients of F -finite regular rings strong F -regularity can be efficiently verified using the following generalization of Fedder's criterion due to Glassbrenner.

Lemma 4.5.13 ([Gla96, Theorem 2.3]). *Let S be a F -finite regular ring such that $F_* S$ is a free S -module and let $R = S/I$. Let s be an element of S not in any minimal prime of I such that R_s is regular. Then, R is strongly F -regular at the maximal ideal \mathfrak{m} if and only if there exists an $e \in \mathbb{N}$ such that $s(I^{[p^e]} : I) \not\subset \mathfrak{m}^{[p^e]}$.*

Again, note that the assumptions of the above lemma are satisfied for $S = k[x_1, \dots, x_n]$ and for S regular local.

Example 4.5.14. The ring $R = k[x_1, \dots, x_n]/(x_1^n + \dots + x_n^n)$ is not strongly F -regular. Indeed, by the above criterion, since $x_1 \in R^\circ$, it suffices to show that $x_1(x_1^n + \dots + x_n^n)^{p-1} \in (x_1^p, \dots, x_n^p)$. If $p \not\equiv 1 \pmod{n}$ then it is straightforward, since in this case for every n integers i_1, \dots, i_n of sum $p-1$ there exists at least one strictly greater than $\frac{p-1}{n}$. Otherwise, we observe that

$$x_1(x_1^n + \dots + x_n^n)^{p-1} \equiv x_1 \cdot (x_1^n \cdots x_n^n)^{\frac{p-1}{n}} \equiv 0 \pmod{x_1^p, \dots, x_n^p}.$$

In the same paper [HH89], the authors relate the strong F -regularity with earlier concepts of F -regularity and weak F -regularity defined using the following notion of tight closure.

Definition 4.5.15. Let I be an ideal in R . We define the *tight closure* of I as an ideal I^* of elements $x \in R$ such that there exists $c \in R^\circ$ satisfying $cx^{p^e} \in I^{[p^e]}$ for every $e \gg 0$. We say that an ideal is *tightly closed* if it is equal to its tight closure.

Example 4.5.16. Let $R = k[x, y, z]/(x^2 + y^3 + z^7)$. It is easy to see that as $k[y, z]$ -module R is free, generated by elements 1 and x . We shall show that $x \notin (y, z)$ but $x \in (y, z)^*$, and thus (y, z) is not tightly closed. First claim is clear. For the second we observe that

$$\begin{aligned} x \cdot x^{p^e} &= (x^2)^{\frac{p^e+1}{2}} = (-y^3 - z^7)^{\frac{p^e+1}{2}} \\ &= (-1)^{\frac{p^e+1}{2}} \left(\sum_{i+j=\frac{p^e+1}{2}} \binom{\frac{p^e+1}{2}}{i} y^{3i} z^{7j} \right). \end{aligned}$$

By a simple direct inspection, we see that either $3i \geq p^e$ or $7j \geq p^e$. This finishes the proof.

Definition 4.5.17. Let R a noetherian ring. We say that R is *weakly F -regular* if every ideal I in R is tightly closed. Moreover, we say that R is *F -regular* if any of its localisations is weakly F -regular.

The peculiar distinction between weakly F -regular, F -regular and strongly F -regular rings is a consequence of the fact that tight closure does not necessarily commute with localization.

Proposition 4.5.18. *Any strongly F -regular ring is F -regular.*

Proof. Since every localisation of a strongly F -regular ring is strongly F -regular (cf. [HH89, Proposition 3.3]) it suffices to show that any ideal $I = (x_1, \dots, x_m)$ in R is tightly closed. Suppose $x \in I^*$. Then there exists $c \in R^\circ$ such that for any $e \gg 0$ we have

$$cx^{p^e} = \sum_i f_i x_i^{p^e},$$

for some elements $f_i \in R$. Since R is strongly F -regular, we see that for sufficiently large e there exists a homomorphism $\phi: F_*^e R \rightarrow R$ satisfying $\phi(c) = 1$. Applying ϕ to the relation above, we see that $x = \sum_i x_i \phi(f_i)$ and therefore $x \in I$. This proves that I is tightly closed. \square

4.5.4 F -rationality and F -injectivity

For the lack of desingularisation and Grauert-Riemenschneider vanishing in characteristic $p > 0$ one attempt to grasp the structure of singularities with properties similar to rational in characteristic 0 using the following notion.

Definition 4.5.19 (F -rationality). Let (R, \mathfrak{m}) be a d -dimensional local ring of characteristic $p > 0$. We say that R is *F -rational* if any ideal generated by a system of parameters is tightly closed.

By the result of Smith [Smi97], we see that the above condition can be expressed in a different manner involving Frobenius action on local cohomology.

Theorem 4.5.20 ([Smi97, Theorem 2.6]). *Let (R, \mathfrak{m}) be a local ring of dimension d . Then R is F -rational if and only if one of the following holds:*

- a) *the ring R is Cohen–Macaulay, and for any $c \in R^\circ$, there exists $e \in \mathbb{N}$ such that $cF^e: H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(R)$ is injective,*
- b) *the ring R is Cohen–Macaulay, and $H_{\mathfrak{m}}^d(R)$ has no proper non-trivial submodules stable under the Frobenius action.*

In the case of \mathbb{N} -graded ring we have the following criterion of Watanabe.

Lemma 4.5.21 (Watanabe a -invariant, cf. [Sin98, Proposition 5.1.1]). *Let (R, \mathfrak{m}) be a Cohen–Macaulay \mathbb{N} -graded ring of dimension d such that the punctured spectrum $\text{Spec}(R) \setminus \{\mathfrak{m}\}$ is F -rational. Then, if the invariant $a(R) \stackrel{\text{def}}{=} \max\{i \in \mathbb{Z} : [H_{\mathfrak{m}}^d(R)]_i \neq 0\}$ satisfies the inequality $a(R) < 0$ then for any $n \gg 0$ the Veronese subring $R^{(n)}$ is F -rational.*

In particular, we see that:

Corollary 4.5.22. *Let X be a smooth projective scheme of dimension $d = \dim X$, and let \mathcal{L} be an ample line bundle. Suppose that $H^d(X, \mathcal{O}_X) = 0$, and for every $j \in \mathbb{Z}$ and every $1 \leq i \leq d - 1$ we have $H^i(X, \mathcal{L}^j) = 0$. Then for sufficiently large n the cone $\text{Cone}_{X, \mathcal{L}^{\otimes n}}$ is F -rational.*

Proof. First, we see that by Prop. 2.5.2 the cone is Cohen-Macaulay. Next, since $H^d(X, \mathcal{O}_X) = 0$, we can use Serre vanishing to prove that for sufficiently large n we have $a(R_{X, \mathcal{L}^{\otimes n}}) < 0$ and therefore we may apply Lemma 4.5.21 to conclude. \square

Example 4.5.23. By the computation in Example 4.5.16 we see that the ring:

$$k[x, y, z]/(x^2 + y^3 + z^7)$$

is not F -rational at (x, y, z) .

Chapter 5

Classical results on mod p^2 and characteristic zero liftability

This chapter contains background results concerning mod p^2 and characteristic zero deformations. First, in §5.1, we present some basic results and give elementary examples. Then, in §5.2, we give a few important results based on the assumption of $W_2(k)$ -liftability. In §5.3, we investigate liftability of the Frobenius morphism. In particular, we present a criterion which substantially restricts the class of Frobenius liftable varieties. We apply it to prove that, except for a few counterexamples, hypersurfaces in \mathbb{P}^n do not lift to $W_2(k)$ compatibly with Frobenius. As it turns out it is not obvious to come up with an example of a scheme which is non-liftable mod p^2 . For this reason, we defer explicit counterexamples to §5.4 and §5.5.

5.1 Preliminary remarks

To lay the groundwork for our future considerations, we give with a few general remarks on mod p^2 and Frobenius liftability. We begin with considerations concerning non-singular and local complete intersection schemes.

5.1.1 $W_2(k)$ -liftability

First, we express Proposition 3.6.7, which is concerned with obstruction theories for deformation functors (cf. Definition 3.6.5), in the setting of smooth schemes.

Proposition 5.1.1. *Let X be a smooth scheme defined over a perfect field k of characteristic $p > 0$. Then the deformation functor Def_X admits an obstruction theory with values in $H^2(X, \mathcal{T}_X)$. Moreover, the tangent space to Def_X is isomorphic to $H^1(X, \mathcal{T}_{X/k})$.*

Proof. Using Theorem 3.1.2 we first observe that $L_{X/k} \simeq \Omega_{X/k}^1$. Then the proof directly follows from Proposition 3.6.7 using the isomorphisms:

$$\mathrm{Ext}^i(\Omega_{X/k}^1, \mathcal{O}_X) \simeq H^i(X, \mathcal{T}_{X/k}), \text{ for } i = 1, 2. \quad \square$$

Corollary 5.1.2. *Smooth affine schemes uniquely lift over every ring in $\mathbf{Art}_{W(k)}(k)$.*

In fact using Theorem 3.1.2 (2) one sees that, except for uniqueness, the same result holds for affine local complete intersection schemes. In the global setting, we have the following proposition which suggests that among projective schemes the class of mod p^2 liftable schemes is ubiquitous.

Proposition 5.1.3. *Let X be a complete intersection inside \mathbb{P}^n defined over a perfect field k . Then for every $A \in \mathbf{Art}_{W(k)}(k)$, in particular $W_2(k)$, there exists a flat A -lifting of X .*

Proof. Let $d = \text{codim}_{\mathbb{P}^n} X$. By assumptions, there exists a sequence of homogeneous elements f_1, \dots, f_d such that $X = V(f_1, \dots, f_d)$. By [Sta17, Tag 00N6], we infer that $\{f_1, \dots, f_d\}$ is a regular sequence. Consequently, using Lemma 2.1.5 and Corollary 2.1.7, we see that for every sequence $\{f'_1, \dots, f'_d\}$ of homogeneous lifts of $\{f_1, \dots, f_d\}$ the scheme $\tilde{X} = V(f'_1, \dots, f'_d)$ is an A -flat lifting of X . This finishes the proof. \square

Remark 5.1.4. Using [Sta17, Tag 0523] and the Grothendieck's existence theorem, the above proof can be adapted to show that X in fact lifts to characteristic zero. This means that the naive philosophy of lifting equations works for complete intersection projective schemes.

5.1.2 Frobenius split schemes are mod p^2 liftable

To illustrate usefulness of functoriality of obstruction classes for deformation problems, we present an existential proof that Frobenius split schemes defined over a perfect field k of characteristic $p > 0$ are mod p^2 liftable. An explicit construction of a lifting of a Frobenius split scheme is contained in §6.3.

Proposition 5.1.5 (B. Bhatt, cf. [Lan16, Proposition 8.4]). *If X/k is a Frobenius split scheme then $X^{(1)}$ lifts to $W_2(k)$.*

Proof. For the case of smooth schemes the reader can see [Ill96, p. 164] or [Jos07, Corollary 9.2]. The proof presented here is due to Bhargav Bhatt. The idea is to use the functoriality of obstructions (see Lemma 3.4.2) for the relative Frobenius $F_{X/k} : X \rightarrow X^{(1)}$. Namely, we have the following commutative diagram with a splitting φ :

$$\begin{array}{ccc} L_{X^{(1)}/k} & \xrightarrow{\sigma_{X^{(1)}}} & \mathcal{O}_{X^{(1)}}[2] \\ \downarrow dF_{X/k} & & \downarrow F^\# \uparrow \varphi \\ F_{X/k*} L_{X/k} & \xrightarrow{F_{X/k*} \sigma_X} & F_{X/k*} \mathcal{O}_X[2], \end{array}$$

where σ_X (resp. $\sigma_{X^{(1)}}$) is the $W_2(k)$ -liftability obstruction of X (resp. $X^{(1)}$). The differential $dF_{X/k} = 0$ and therefore by existence of the splitting $\sigma_{X^{(1)}} = 0$. \square

Note that in case of varieties over a perfect field the $W_2(k)$ -liftability of $X^{(1)}$ is in fact equivalent to the liftability of X (by untwisting using the inverse of Frobenius).

5.1.3 Frobenius liftability

Trying to apply techniques of functors of Artin rings to deformations of the Frobenius morphism, one encounters the problem that flat A -lifts of the Frobenius morphism of a given scheme X are not morphisms over the base, but only commute with chosen Frobenius lift of A . Such lifts are canonically defined only for the rings of Witt vectors, and therefore a suitable base category consists of pairs (A, F_A) , where A is a ring and $F_A : A \rightarrow A$ is a lift of Frobenius. Unfortunately, this category falls outside of the scope of standard results in deformation theory. Leaving this problem aside, we focus on $W_2(k)$ -deformations.

Proposition 5.1.6. *Let X be a smooth scheme defined over a perfect field k of characteristic $p > 0$, and let $X \rightarrow \tilde{X}$ be a deformation mod p^2 . Then the obstruction class for existence of a compatible lifting of Frobenius lies in $H^1(X, F^*\mathcal{T}_X)$. In case the obstruction class vanishes, the set of liftings is a torsor under $H^0(X, F^*\mathcal{T}_X)$.*

Proof. Using Theorem 3.1.2, we see that $L_{X/k} \simeq \Omega_{X/k}^1$, and therefore the claim follows from Lemma 3.7.2 and the isomorphisms $\mathrm{Ext}^i(F_{X/k}^* \Omega_{X^{(1)}/k}^1, \mathcal{O}_X) \simeq H^i(X, F^*\mathcal{T}_X)$, for $i = 0, 1$. \square

Corollary 5.1.7. *Smooth affine schemes admit (not necessarily unique) lifts of the Frobenius morphism over $W_2(k)$.*

In this direction, in §7.2, we develop an explicit criterion for Frobenius liftability of affine complete intersection schemes. In the global case, unlike mod p^2 liftability, it is difficult to recognize a large class of examples of Frobenius liftable schemes. A remarkable one is the class of toric varieties.

Example 5.1.8 (Frobenius liftability of toric varieties). Every (normal separated) toric variety X defined over a perfect field k of characteristic $p > 0$ is Frobenius liftable.

Proof. It is well-known (see [Ful93, Section 1.4]) that X is covered by sets U_i isomorphic to spectra of a k -algebras $k[M_i]$ associated to finitely generated monoids M_i induced by cones in the fan of the variety. The sets U_i admit Frobenius liftings given by $W_2(k)$ -algebras $W_2(k)[M_i]$ and the corresponding Frobenius induced by the monoid mapping $M_i \ni m \mapsto pm$. This choice of local lifting glues to give a global Frobenius lifting of the variety X . \square

One of the reasons for scarcity of globally Frobenius liftable schemes is the following result. In particular, it implies that there are no Frobenius liftable smooth schemes of positive Kodaira dimension.

Proposition 5.1.9 ([DI87, Théorème 2.1 (b)], [Xin16, Proposition 2.8]). *If X is a smooth Frobenius liftable scheme over a perfect field k then there exists an injective morphism*

$$F_{X/k}^* \Omega_{X^{(1)}/k}^1 \rightarrow \Omega_{X/k}^1.$$

Remark 5.1.10. Let us rephrase the above statement in the setting of the absolute Frobenius morphism. Using base change isomorphism, we see that $\Omega_{X^{(1)}/k}^1 \simeq W_{X/k}^* \Omega_{X/k}^1$. Therefore we obtain an injective morphism (cf. §1.4):

$$F^* \Omega_{X/k}^1 \simeq F_{X/k}^* W_{X/k}^* \Omega_{X/k}^1 \simeq F_{X/k}^* \Omega_{X^{(1)}/k}^1 \rightarrow \Omega_{X/k}^1.$$

5.2 Consequences of mod p^2 liftability

5.2.1 Decomposition of the de Rham complex and Hodge–de Rham spectral sequence

Theorem 5.2.1 ([DI87, Théorème 2.1]). *Let X be a smooth scheme over a perfect field k of characteristic $p > 0$. Suppose \tilde{X} is a $W_2(k)$ -flat lifting of X . Then there exists a quasi-isomorphism:*

$$\phi_{\tilde{X}}: \bigoplus_{i < p} \Omega_{X^{(1)}/k}^i[-i] \simeq \tau_{< p} F_{X/k*} \Omega_{X/k}^\bullet,$$

inducing the inverse Cartier isomorphism (see Theorem 4.3.1) on cohomology sheaves.

Remark 5.2.2. Instead of reproducing the whole proof, we refer to Theorem 5.3.1 for the proof in the special case when \tilde{X} admits a compatible lifting of the Frobenius morphism $\tilde{F}: \tilde{X} \rightarrow \tilde{X}$. The more general case follows by a glueing procedure based on Čech resolution, which reduces the problem to the mentioned one.

As a simple corollary we obtain the degeneration of Hodge–de Rham spectral sequence (cf. §4.2) for varieties which can be lifted to $W_2(k)$.

Corollary 5.2.3 ([DI87, Corollaire 2.4]). *Let X be a smooth scheme over a perfect field k of characteristic $p > 0$. Suppose that $\dim X < p$, and that X admits a $W_2(k)$ -lifting. Then the Hodge–de Rham and conjugate spectral sequences degenerate at the first page.*

Remark 5.2.4. Using standard techniques, the above result also leads to a simple, algebraic proof of Kodaira–Akizuki–Nakano vanishing.

5.2.2 Ogus–Vologodsky and Bogomolov–Miyaoka–Yau inequality

Under the assumption of $W_2(k)$ -liftability Ogus and Vologodsky (see [OV07]) constructed equivalence between certain categories of vector bundles equipped with additional structures (Higgs fields and integrable connections satisfying nilpotence conditions). Their construction was based on a careful analysis of the torsor of local Frobenius liftings. The correspondence was then applied by Langer to prove the following characteristic p version of Bogomolov–Miyaoka–Yau inequality.

Theorem 5.2.5 ([Lan15, Theorem 13]). *Let X be a smooth projective surface of non-negative Kodaira dimension. Assume that X can be lifted to $W_2(k)$. If $p \geq 3$ then $3c_2(X) \geq c_1^2(X)$.*

Remark 5.2.6. As we outlined in the introduction, one can prove, using flat invariance of intersection numbers, that the same assertion holds if X lifts to characteristic zero.

5.3 Further remarks on Frobenius liftability

Here we present some further results concerning Frobenius liftability. We begin by showing that it has certain cohomological implications, which lead to partial classification results.

5.3.1 Results of Buch–Thomsen–Lauritzen–Mehta

In [BTLM97] one finds the following strengthening of the result of Deligne and Illusie.

Lemma 5.3.1 ([BTLM97, Theorem 2]). *Suppose X is a smooth Frobenius liftable scheme over a perfect field k of characteristic $p > 0$. Then there is a split quasi-isomorphism of sheaves of differential graded algebras $\bigoplus_i \Omega_{X/k}^i[-i] \rightarrow F_* \Omega_{X/k}^\bullet$, which induces Cartier isomorphism on cohomology.*

As a corollary the authors obtain the following vanishing theorem for Frobenius liftable varieties.

Definition 5.3.2. We say that a projective variety X/k satisfies *Bott vanishing* if for any ample line bundle \mathcal{L} on X we have:

$$H^i(X, \Omega_{X/k}^j \otimes \mathcal{L}) = 0 \quad \text{for any } i > 0 \text{ and } j \geq 0.$$

Remark 5.3.3. Using Serre vanishing we see that Bott vanishing is equivalent to the vanishing:

$$H^i(X, \Omega_{X/k}^j \otimes \mathcal{L}^{-1}) = 0 \quad \text{for any } i < \dim X \text{ and } j \geq 0,$$

for every ample line bundle \mathcal{L} on X .

Theorem 5.3.4 ([BTLM97, Theorem 3]). *Let X/k be a smooth Frobenius liftable variety. Then X satisfy Bott vanishing.*

Proof. By Lemma 5.3.1, for any $j \geq 0$ we obtain a split monomorphism $\Omega_{X/k}^j \hookrightarrow F_* \Omega_{X/k}^j$. After twisting with an ample line bundle \mathcal{L} and using projection formula we get a split monomorphism $\Omega_{X/k}^j \otimes \mathcal{L} \hookrightarrow F_*(\Omega_{X/k}^j \otimes \mathcal{L}^p)$. By taking cohomology, for any $i > 0$ we obtain an injection $H^i(X, \Omega_{X/k}^j \otimes \mathcal{L}) \hookrightarrow H^i(X, \Omega_{X/k}^j \otimes \mathcal{L}^p)$, which by inductive arguments implies that $H^i(X, \Omega_{X/k}^j \otimes \mathcal{L}) \hookrightarrow H^i(X, \Omega_{X/k}^j \otimes \mathcal{L}^{p^n})$ for any $n \geq 1$. This finishes the proof because by Serre vanishing $H^i(X, \Omega_{X/k}^j \otimes \mathcal{L}^{p^n}) = 0$ for n large enough \square

Corollary 5.3.5. *Every Frobenius liftable Fano variety X of dimension $\dim X \geq 2$ is rigid, that is, $H^1(X, \mathcal{T}_X) = 0$.*

Proof. Since ω_X is anti-ample, using Serre duality and Remark 5.3.3, we see that $H^1(X, \mathcal{T}_X) = H^{\dim X - 1}(X, \Omega_{X/k}^1 \otimes \omega_X) = 0$. \square

This leads to the following classifying result. In the proof we follow [Xin16], where the case of surfaces is considered.

Proposition 5.3.6 (Extension of [Xin16]). *Let X be a hypersurface in \mathbb{P}^n , for $n \geq 4$, of degree $d \geq 3$. Then X is not Frobenius liftable.*

Proof. Suppose X is Frobenius liftable. Let d be the degree of X . First, using adjunction formula, we observe that the canonical line bundle ω_X is isomorphic to $\mathcal{O}_X(d - n - 1)$. By Proposition 5.1.9 we see that there exists an injective morphism $\psi: F^* \Omega_{X/k}^1 \rightarrow \Omega_{X/k}^1$, which induces an injective determinant map $\bigwedge^n \psi: \omega_X^{\otimes p} \simeq F^* \omega_X \rightarrow \omega_X$. If $d > n + 1$ this gives a contradiction.

If $d = n + 1$ then the determinant map is a non-zero endomorphism of a trivial line bundle and thus an isomorphism. This implies that ψ itself is an isomorphism, and thus $\Omega_{X/k}^1$ is Frobenius stable (we say that \mathcal{E} is Frobenius stable if $F^* \mathcal{E} \simeq \mathcal{E}$). By [LS77, 1.4 Satz], the cotangent bundle is therefore étale trivialisable and hence trivial (using Lefschetz for the fundamental group $\pi_1(X)$). However, since $H^1(X, \mathcal{O}_X) = 0$, the long exact sequences associated with the short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \Omega_{\mathbb{P}^n|X}^1 \longrightarrow \Omega_{X/k}^1 \simeq \mathcal{O}_X^{n-1} \longrightarrow 0 \quad (5.1)$$

and the Euler sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^n|X}^1 \longrightarrow \mathcal{O}_X(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0, \quad (5.2)$$

imply that

$$\begin{aligned} n - 1 &= \dim_k H^0(X, \mathcal{O}_X^{\oplus n-1}) \\ &= \dim_k H^0(X, \Omega_{\mathbb{P}^n|X}^1) \\ &\leq \dim_k H^0(X, \mathcal{O}_X(-1)^{\oplus n+1}) = 0, \end{aligned}$$

which gives a contradiction. Finally, if $d < n + 1$ then X is Fano. Using Corollary 5.3.5, we therefore see that X is rigid. However, from the long exact sequences associated with the duals of (5.1) and (5.2), we obtain a diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& H^0(X, \mathcal{O}_X(1)^{\oplus n+1}) & & & H^1(X, \mathcal{O}_X(1)^{\oplus n+1}) = 0 & & \\
& \downarrow & & & \downarrow & & \\
\cdots & \longrightarrow H^0(X, \mathcal{T}_{\mathbb{P}^n|X}) & \longrightarrow & H^0(X, \mathcal{O}_X(d)) & \longrightarrow & H^1(X, \mathcal{T}_X) & \longrightarrow \cdots \\
& \downarrow & & & & \downarrow & \\
& H^1(X, \mathcal{O}_X) = 0 & & & & H^2(X, \mathcal{O}_X) = 0, &
\end{array}$$

Since $d \geq 3$ and $n + d \geq 7$, this implies that (the hypersurfaces in question are projectively normal)

$$\begin{aligned}
\dim_k H^1(X, \mathcal{T}_X) &\geq \dim H^0(X, \mathcal{O}_X(d)) - \dim_k H^0(X, \mathcal{T}_{\mathbb{P}^n|X}) \\
&\geq \left(\binom{n+d}{d} - 1 \right) - ((n+1)^2 - 1) \\
&= \binom{n+d}{3} - (n+1)^2 \\
&= \frac{n+d}{6} \cdot (n+d-1)(n+d-2) - (n+1)^2 > 0. \quad \square
\end{aligned}$$

Remark 5.3.7. Unsurprisingly, the estimates in the second line of the above computation correspond to the dimension of the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$, and the dimension of the group $\mathbb{P}\mathrm{GL}_N$, that is, the automorphism group of \mathbb{P}^n .

The Bott vanishing was used in [BTLM97, Theorem 6] to give a partial classification of Frobenius liftable projective homogeneous spaces.

Theorem 5.3.8 ([BTLM97]). *Let G be a semi-simple algebraic group defined over a perfect field k of characteristic p , and let P be a parabolic subgroup. Assume that either G is of type A and G/P is not a projective space, or that P is contained in a maximal parabolic subgroup as listed in [BTLM97, 4.3.1–4.3.7]. Then the homogeneous space G/P does not admit a $W_2(k)$ -lifting compatible with Frobenius.*

Remark 5.3.9. The class of homogeneous spaces satisfying the assumptions of the above theorem contains in particular Grassmannians $\mathrm{Gr}(n, k)$ ($1 < k < n-1$), full flag varieties SL_n/B ($n \geq 3$, B = upper-triangular matrices), or smooth quadric hypersurfaces in \mathbb{P}^n , $n \geq 4$. Presumably all homogeneous spaces which are not toric (i.e., not a product of projective spaces) do not admit a lift to $W_2(k)$ together with Frobenius.

In a similar direction, we have the following result of Paranjape and Srinivas.

Theorem 5.3.10 ([PS89, Theorem]). *Let G be a semi-simple, simply connected algebraic group over an algebraically closed field k . Let X be a homogeneous space for G . Then X admits a characteristic zero lifting compatible with Frobenius morphism if and only if $X \simeq \prod \mathbb{P}^{n_i}$, for some integers n_i .*

5.3.2 Local Frobenius liftability and finite-generation of crystalline cohomology

In [Bha12], Bhatt generalised the results of Deligne and Illusie (cf. Theorem 5.2.1) to the singular context. More precisely, he shows that a certain derived version of the de

Rham complex decomposes as a direct sum of exterior powers of the cotangent complex. Using this results together with a comparison theorem between derived de Rham and crystalline cohomology he proves the following:

Theorem 5.3.11 ([Bha14, Proposition 3.1]). *Let X be a proper local complete intersection scheme over k . Assume there exists an isolated singular closed point $x \in X$ such that there exists a Frobenius lifting of the neighbourhood of x . Set $N = \dim_{k(x)} \mathfrak{m}_x / \mathfrak{m}_x^2$. Then there exists an integer $0 < i \leq N$ such that:*

- 1) $H_{crys}^N(X/k)$ is infinitely generated over k ,
- 2) $H_{crys}^{N-i}(X/k)$ is infinitely generated over k ,
- 3) At least one of $H_{crys}^{N+1}(X/W)[p]$ and $H_{crys}^N(X/W)/p$ is infinitely generated over k ,
- 4) At least one of $H_{crys}^{N+1-i}(X/W)[p]$ and $H_{crys}^{N-i}(X/W)/p$ is infinitely generated over k .

5.4 Classical examples of non-liftable schemes

In this section we present two classical results concerning non-liftability of schemes. In the first case we only give a general overview of constructions and proofs. The second example, described in details, serves as an evidence of subtlety of the matter.

5.4.1 Serre's example

The first example of a non-liftable scheme appeared in [Ser61]. It is given as a quotient of a hypersurface in \mathbb{P}^n , for $n \geq 4$, by a carefully chosen p -group $G \subset \mathbb{PGL}_n$.

Proposition 5.4.1 ([FGI⁺05, Corollary 8.6.7, Section 8.7]). *Let r, n be integers such that $2 < r < n$, and let p be a prime satisfying $p > n + 1$. Let $G = \mathbb{F}_p^s$, with $s > n + 1$. There exists an action of G on \mathbb{P}^n , and a smooth, complete intersection Y_0 of dimension r in \mathbb{P}^n , stable under the action of G and on which G acts freely, and such that the smooth, projective scheme $X_0 = Y_0/G$ has the following property. Let A be a noetherian ring with residue field k and such that $p \neq 0$. Then there exists no scheme X , flat over A , lifting X_0 . In particular, X does not lift neither over $W_2(k)$ nor to characteristic zero.*

Remark 5.4.2. The above result provides a counterexample to Lemma 3.7.1, if the degree of the morphism π is divisible by p .

5.4.2 Examples arising from Kodaira fibrations

We shall now present an example of a surface of general type violating Bogomolov-Miyaoka-Yau inequality and thus non-liftable neither to characteristic 0 nor to $W_2(k)$ (cf. Theorem 5.2.5 and Remark 5.2.6). The idea behind the example and most of the considerations are taken from [Szp81, Exposé III & IV]. The construction is based on existence of so-called Kodaira fibrations.

Definition 5.4.3 (Kodaira fibration). We say that a morphism $f: X \rightarrow C$ is a *Kodaira fibration* if X is a smooth projective surface, C is a smooth projective curve of genus $g_C \geq 2$ and f is a smooth non-isotrivial morphism with genus of the fibres equal to $g_F \geq 2$.

Given a Kodaira fibration, we obtain a non-liftable scheme by the following:

Theorem 5.4.4. *Suppose $f: X \rightarrow C$ is a Kodaira fibration. Then for $n \in \mathbb{N}$ sufficiently large the surface $X^{(n/C)} = X \times_{F_C^n, C} C$ violates the Bogomolov-Miyaoka-Yau inequality, and therefore lifts neither to characteristic 0 nor to $W_2(k)$.*

Proof. We begin by observing that by the Noether's formula for any surface S we have:

$$\chi(\mathcal{O}_S) = \frac{c_1(S)^2 + c_2(S)}{12}$$

and therefore the Bogomolov-Miyaoka-Yau inequality $3c_2(S) \geq c_1^2(S)$ is equivalent to:

$$c_2(S) \geq 3\chi(\mathcal{O}_S).$$

In case of the surface $X^{(n/C)}$ the components of the above inequality can be expressed as follows. By definition $X^{(n/C)}$ arises from the cartesian diagram of relative Frobenius:

$$\begin{array}{ccc} X^{(n/C)} & \xrightarrow{W_{X/C}} & X \\ \downarrow f^{(n)} & & \downarrow f \\ C & \xrightarrow{F_S^n} & C, \end{array}$$

and consequently admits a smooth fibration $f^{(n)} : X^{(n/C)} \rightarrow C$. Using the relative cotangent sequence:

$$0 \longrightarrow f^{(n)*}\Omega_C^1 \longrightarrow \Omega_{X^{(n/C)}}^1 \longrightarrow \Omega_{X^{(n/C)}/C}^1 \longrightarrow 0,$$

we compute $c_2(X^{(n/C)})$ as follows:

$$\begin{aligned} c_2(X^{(n/C)}) &= (f^{(n)*}c_1(C)) \cdot c_1(\Omega_{X^{(n/C)}}^1) \\ &= (2g_C - 2)F \cdot c_1(\Omega_{X^{(n/C)}}^1) \\ &= (2g_C - 2) \deg \Omega_{X^{(n/C)}/C}^1 \\ &= (2g_C - 2) \deg \Omega_F^1 = (2g_C - 2)(2g_F - 2) = 4(g_C - 1)(g_F - 1). \end{aligned}$$

Leray spectral sequence for morphism $f^{(n)}$ of relative dimension 1 together with additivity of the Euler characteristic yields the equality:

$$\chi(\mathcal{O}_{X^{(n/C)}}) = \chi(f_*^{(n)}\mathcal{O}_{X^{(n/C)}}) - \chi(R^1f_*^{(n)}\mathcal{O}_{X^{(n/C)}}).$$

The direct image $f_*^{(n)}\mathcal{O}_{X^{(n/C)}}$ is isomorphic to \mathcal{O}_C because the fibres of $f^{(n)}$ are connected, and $R^1f_*^{(n)}\mathcal{O}_{X^{(n/C)}}$ is isomorphic to $F_S^{n*}R^1f_*\mathcal{O}_X$ by flat base change applied to the cartesian square of Frobenius.

Using the above reformulation implied by the Noether's formula, the Bogomolov-Miyaoka-Yau inequality can be written as:

$$\begin{aligned} 4(g_C - 1)(g_F - 1) = c_2(X^{(n/C)}) &\geq 3\chi(\mathcal{O}_{c_2(X^{(n/C)})}) \\ &= 3(\chi(\mathcal{O}_C) - \chi(F_C^{n*}R^1f_*\mathcal{O}_X)) \\ &= 3(1 - g_C - \deg(F_C^{n*}R^1f_*\mathcal{O}_X) - g_F(1 - g_C)) \\ &= 3(g_C - 1)(g_F - 1) - 3\deg(F_C^{n*}R^1f_*\mathcal{O}_X), \end{aligned}$$

where in penultimate line we used the Riemann-Roch theorem. The above inequality is equivalent to:

$$(g_C - 1)(g_F - 1) \geq -3 \deg(F_C^{n*} R^1 f_* \mathcal{O}_X) = -3p^n \cdot \deg(R^1 f_* \mathcal{O}_X),$$

which is clearly a contradiction for $n \in \mathbb{N}$ sufficiently large, if the degree of $R^1 f_* \mathcal{O}_X$ is negative. By relative Serre duality $R^1 f_* \mathcal{O}_X \simeq (f_* \omega_{X/C})^\vee$ and therefore the last statement follows from positivity of Hodge bundles (see Theorem 5.4.5 below). \square

For the sake of completeness, we now address the problems of existence of Kodaira fibrations and positivity of Hodge bundles.

Positivity of Hodge bundles and existence of Kodaira fibrations

Theorem 5.4.5 ([Szp81, Exposé III]). *Let $f: X \rightarrow C$ be a Kodaira fibration (or more generally a semi-stable fibration). Then $\deg(f_* \omega_{X/C}) > 0$.*

We now proceed to the proof of existence of Kodaira fibrations. In view of the beauty of the argument and inaccessibility of Asterisques, we give a detailed proof.

Theorem 5.4.6 ([Szp81, Exposé IV]). *There exists a Kodaira fibration defined over an algebraically closed field k of any characteristic.*

Proof. We closely follow the approach presented in [Szp81, Exposé IV]. We fix an integer n relatively prime to the characteristic k .

We begin by taking an étale covering $f: C_1 \rightarrow C_0$ of degree n of smooth curves over k of genera $g_{C_1} \geq 2$ and $g_{C_0} \geq 2$, respectively. We denote by Γ_f a divisor on $C_0 \times C_1$ associated to the graph of f and by $q_0: C_0 \times C_1 \rightarrow C_0$ the natural projection. We claim that there exists an étale covering $\pi: C \rightarrow C_0$ such that the line bundle $(\pi \times \text{id})^* \mathcal{O}_{C_0 \times C_1}(\Gamma_f)$ is isomorphic to the n -th power of a line bundle \mathcal{R} , and that the associated n -fold cyclic covering ramified over $(\pi \times \text{id})^{-1} \Gamma_f$ admits the structure of a Kodaira fibration.

To prove the first claim we take a k -rational point $s \in C_1$ and consider the line bundle $\mathcal{L} = \mathcal{O}_{C_0 \times C_1}(\Gamma_f - n \cdot S)$, where $S = C_0 \times \{s\}$. For any $c \in C_0$ the restriction $\mathcal{L}|_{c \times C_1}$ is of degree 0 and therefore \mathcal{L} constitutes a family of degree 0 line bundles on C_1 . This induces a classifying morphism $f: C_0 \rightarrow \text{Pic}^0(C_1)$ such that $\mathcal{L} \simeq (f \times \text{id})^* \mathcal{P} \otimes q_0^* \mathcal{M}$ where \mathcal{P} is a Poincaré line bundle on $\text{Pic}^0(C_1) \times C_1$ and \mathcal{M} is a uniquely defined line bundle on C_0 . We take $\pi: C \rightarrow C_0$ to be the étale covering of C_0 defined by the cartesian diagram:

$$\begin{array}{ccc} C & \xrightarrow{\tilde{f}} & \text{Pic}^0(C_1) \\ \downarrow \pi & & \downarrow n \cdot \\ C_0 & \xrightarrow{f} & \text{Pic}^0(C_1), \end{array}$$

and set $q: C \times C_1 \rightarrow C$ to be the projection. The pullback of \mathcal{L} under $\pi \times \text{id}: C \times C_1 \rightarrow C_0 \times C_1$ satisfies the relation:

$$(\pi \times \text{id})^* \mathcal{L} \simeq ((\pi \circ f) \times \text{id})^* \mathcal{P} \otimes (\pi \times \text{id})^* q_0^* \mathcal{M} \simeq (\tilde{f} \times \text{id})^* \mathcal{P}^n \otimes q^* \pi^* \mathcal{M},$$

where the second isomorphism follows from the definition of C and moduli interpretation of the multiplication map $n \cdot: \text{Pic}^0(C_1) \rightarrow \text{Pic}^0(C_1)$. The degree of π is equal to $n^{2g_{C_0}}$

which implies that $d = \deg(\pi^*\mathcal{M})$ is divisible by n . The field k is algebraically closed and therefore the multiplication by n mapping:

$$n \cdot : \text{Pic}^{d/n}(C) \rightarrow \text{Pic}^d(C)$$

is surjective on k -points and thus there exists a line bundle \mathcal{M}' on C such that $\pi^*\mathcal{M} = \mathcal{M}'^{\otimes n}$. As a result we see that:

$$(\pi \times \text{id})^* \mathcal{O}_{C_0 \times C_1}(\Gamma_f) = \left((\tilde{f} \times \text{id})^* \mathcal{P} \otimes q^* \mathcal{M}' \otimes (\pi \times \text{id})^* \mathcal{O}_{C_0 \times C_1}(S) \right)^{\otimes n},$$

and therefore we may set $\mathcal{R} = (\tilde{f} \times \text{id})^* \mathcal{P} \otimes q^* \mathcal{M}' \otimes (\pi \times \text{id})^* \mathcal{O}_{C_0 \times C_1}(S)$. This ends the proof of the first claim.

To prove the second claim we set $g : Y \rightarrow C \times C_1$ to be the n -fold cyclic covering associated to a section $\Gamma'_f = (\pi \times \text{id})^{-1} \Gamma_f$ of $\mathcal{R}^{\otimes n}$. We claim that Y is smooth and that the composition $q \circ g : Y \rightarrow C$ equips it with a structure of non-isotrivial smooth fibration. The divisor Γ'_f is smooth which implies that Y is smooth. For a given point $c \in C$ the fibre $F_c = (q \circ g)^{-1}(c)$ is isomorphic to an n -fold covering of C_1 over a reduced subscheme $f^{-1}(\pi(c))$ which is smooth, and therefore $q \circ g$ is a smooth morphism.

To finish the proof we assume, for the sake of contradiction, that $q \circ g$ is iso-trivial. The fibres F_c are therefore isomorphic to a given curve C' and consequently, for any $c_0 \in C_0$ we obtain a separable ramified covering $g_{c_0} : C' \rightarrow C$ defined by the diagram:

$$\begin{array}{ccccc} & & g_{c_0} & & \\ & \nearrow & & \searrow & \\ C' & \xrightarrow{\cong} & F_c & \xrightarrow{g|_{F_c}} & C_1 \times C \xrightarrow{p_1} C_1, \end{array}$$

where $c \in C$ satisfies $\pi(c) = c_0$ and $p_1 : C \times C_1 \rightarrow C_1$ is the projection. The ramification loci of morphisms g_{c_0} vary for different choices of $c_0 \in C_0$ and therefore we obtain a contradiction with:

Theorem 5.4.7 (de Franchis). *Let C' and C be two curves over a field k of genera $g_C \geq 2$ and $g_{C'} \geq 2$, respectively. Then there exist only finitely many separable morphisms $C' \rightarrow C$.*

Proof. We present a sketch of a proof based on the theory of Hom-schemes (see, e.g., [Kol96, Section II.1]). Firstly, we show that the degree of a map $f : C' \rightarrow C$ is bounded from above by a function depending solely on g_C and $g_{C'}$. Indeed, using Riemann–Hurwitz formula we see that:

$$2g_{C'} - 2 = \deg(f)(2g_C - 2) + \text{len}(R_f),$$

where R_f is the ramification divisor of f . This clearly implies that $\deg(f) \leq d_{C,C'} \stackrel{\text{def}}{=} \frac{2g_{C'} - 2}{2g_C - 2}$. As a consequence, we see that the morphisms $C' \rightarrow C$ arise as k -points of a finite type k -scheme:

$$\text{Hom}^{d_{C,C'}}(C', C)$$

of morphisms from C' to C of degree bounded by $d_{C,C'}$, and therefore to finish the proof it suffices to show that for any $f : C' \rightarrow C$ the tangent space $T_{[f]} \text{Hom}(C', C)$ is zero-dimensional. For this purpose, we apply [Kol96, Section II.1, Theorem 1.7] to identify the tangent space $T_{[f]} \text{Hom}(C', C)$ with $H^0(C', f^* \mathcal{T}_C)$, which is 0 by ampleness of $\Omega_C^1 \simeq \mathcal{T}_C^\vee$. \square

As presented above, this ends the proof of existence of Kodaira fibrations. \square

5.5 Mod p^2 and Frobenius liftability of singular schemes

Proposition 5.5.1. *Let X be a smooth projective scheme which does not lift mod p^2 . Then for sufficiently ample line bundle \mathcal{L} the cone $\text{Cone}_{X,\mathcal{L}}$ does not lift to mod p^2 either.*

Proof. This follows directly from Proposition 3.6.19 □

Chapter 6

Mod p^2 and characteristic 0 liftability of schemes

In this chapter we present our main results concerning mod p^2 liftability of schemes. The content is organized as follows. First, in §6.1 we give an example of a non-liftable zero-dimensional scheme. Then, in §6.2 we provide a few new examples of smooth schemes that are non-liftable modulo p^2 and non-liftable to characteristic zero. These examples arise from schemes which do not admit a $W_2(k)$ -lifting compatible with Frobenius and from interesting configurations of lines in projective spaces over \mathbb{F}_p . In §6.3 we give an explicit construction of a $W_2(k)$ -lifting of a Frobenius split scheme. Finally, in §6.4 for a given family of schemes, we investigate the structure of the set of closed points in the base such that the fibre is liftable mod p^2 .

6.1 Example of non $W_2(k)$ -liftable scheme, p -neighbourhoods of smooth quadrics

As we remarked it is difficult to find either projective or affine non-liftable schemes. In this section, we present a zero-dimensional example. It first appeared in [BO78, Section 3.4] (the authors attribute it to Koblitz) as an instance of a different phenomenon and was only mentioned in our context (without a proof) in the recent paper [Bha12, Remark 3.16]. We give a direct computational proof. Later we use it to study Frobenius liftability of ordinary double points. More precisely, we show that the Artinian local k -algebras:

$$(A_{2n-1}, \mathfrak{m}_{2n-1}) = \left(k[x_1, \dots, x_{2n-1}] / (x_1x_2 + x_3x_4 + \dots + x_{2n-3}x_{2n-2} + x_{2n-1}^2, x_1^p, \dots, x_{2n-1}^p), (x_1, \dots, x_{2n-1}) \right) \\ (A_{2n}, \mathfrak{m}_{2n}) = \left(k[x_1, \dots, x_{2n}] / (x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n}, x_1^p, \dots, x_{2n}^p), (x_1, \dots, x_{2n}) \right)$$

are non-liftable to $W_2(k)$ for $n \geq 3$.

Proposition 6.1.1. *For any $N \geq 5$ the local algebra A_N does not admit a $W_2(k)$ -lifting.*

Proof. Let us set

$$f_N = \begin{cases} x_1x_2 + x_3x_4 + \dots + x_{2n-3}x_{2n-2} + x_{2n-1}^2, & \text{if } N = 2n - 1 \\ x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n}, & \text{if } N = 2n \end{cases}$$

First, let us consider the case when N is even, that is, $N = 2n$. Suppose, to derive a contradiction, that A_{2n} admits a flat $W_2(k)$ -lifting. By Lemma 2.1.8, there exists a

choice of liftings $\widetilde{f_{2n}}, \widetilde{x_1^p}, \dots, \widetilde{x_{2n}^p}$ of $f_{2n}, x_1^p, \dots, x_{2n}^p$, such that the ring

$$\widetilde{A} = W_2[x_1, \dots, x_{2n}] / (\widetilde{f_{2n}}, \widetilde{x_1^p}, \dots, \widetilde{x_{2n}^p})$$

is flat over $W_2(k)$. By means of Corollary 2.1.7 we see that

$$\widetilde{C} = W_2[x_1, \dots, x_{2n}] / (\widetilde{x_1^p}, \dots, \widetilde{x_{2n}^p})$$

is a flat $W_2(k)$ -lifting of $C \stackrel{\text{def}}{=} k[x_1, \dots, x_{2n}] / (x_1^p, \dots, x_{2n}^p) \simeq \widetilde{C} \otimes_{W_2(k)} k$. Consequently we consider the following commutative diagram obtained by applying the functors $\otimes_{W_2(k)} \widetilde{A}$ and $\otimes_{W_2(k)} \widetilde{C}$ to the short exact sequence $0 \rightarrow pW_2(k) \rightarrow W_2(k) \rightarrow k \rightarrow 0$:

$$\begin{array}{ccccccc} & & & \text{Ann}_{\widetilde{C}}(\widetilde{f_{2n}}) & \longrightarrow & \text{Ann}_C(f_{2n}) & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & pW_2(k) \otimes \widetilde{C} \simeq C & \xrightarrow{\cdot p} & \widetilde{C} & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow \cdot \widetilde{f_{2n}} & & \downarrow \cdot \widetilde{f_{2n}} & & \downarrow \\ 0 & \longrightarrow & pW_2(k) \otimes \widetilde{C} \simeq C & \xrightarrow{\cdot p} & \widetilde{C} & \longrightarrow & \widetilde{C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & pW_2(k) \otimes \widetilde{A} \simeq A & \xrightarrow{\cdot p} & \widetilde{A} & \longrightarrow & A. \end{array}$$

Since \widetilde{A} is $W_2(k)$ -flat, we see that the canonical homomorphism $\text{Ann}_{\widetilde{C}}(\widetilde{f_{2n}}) \rightarrow \text{Ann}_C(f_{2n})$ coming from the snake lemma is surjective. In particular,

there exists an element $\widetilde{f_{2n}^{p-1}} + pg \in \text{Ann}_{\widetilde{C}}(\widetilde{f_{2n}})$ lifting f_{2n}^{p-1} . Such element satisfies the relation $(\widetilde{f_{2n}^{p-1}} + pg)\widetilde{f_{2n}} = \widetilde{f_{2n}^p} + pgf_{2n} = 0 \in \widetilde{C}$. By direct computation we see that

$$\begin{aligned} \widetilde{f_{2n}^p} &= (x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n} + pu)^p \\ &= (x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n})^p \\ &= x_1^p x_2^p + x_3^p x_4^p + \dots + x_{2n-1}^p x_{2n}^p + pP_{f_{2n}} = pP_{f_{2n}}, \end{aligned}$$

where

$$P_{f_{2n}} = \sum_{\substack{i_1 + \dots + i_n = p \\ i_1, \dots, i_n \neq p}} \frac{(p-1)!}{i_1! \dots i_n!} (x_1x_2)^{i_1} (x_3x_4)^{i_2} \dots (x_{2n-1}x_{2n})^{i_n} \in \widetilde{C} / p\widetilde{C}.$$

Therefore, we have $p(P_{f_{2n}} + gf_{2n}) = 0$ in \widetilde{C} , which by Corollary 2.1.10 means that:

$$P_{f_{2n}} \in (f_{2n}, x_1^p, \dots, x_{2n}^p). \quad (6.1)$$

Considering the quotient map

$$k[x_1, \dots, x_{2n}] \rightarrow k[x_1, \dots, x_{2n}] / (x_7, \dots, x_{2n}) \simeq k[x_1, \dots, x_6],$$

we see that $P_{f_6} \in (f_6, x_1^p, \dots, x_6^p)$. Now, we claim that $(x_5x_6)^{p-2}P_{f_6} = -(x_3x_4x_5x_6)^{p-1} \pmod{f_6, x_1^p, \dots, x_6^p}$. Indeed, this follows from the computation:

$$\begin{aligned}
(x_5x_6)^{p-2}P_{f_6} &= (x_5x_6)^{p-2} \sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{(p-1)!}{i!j!k!} (x_1x_2)^i (x_3x_4)^j (x_5x_6)^k \\
&= (x_5x_6)^{p-2} \sum_{\substack{i+j=p \\ i,j \neq p}} \frac{(p-1)!}{i!j!} (-1)^i (x_3x_4 + x_5x_6)^i (x_3x_4)^j \\
&\quad + (x_5x_6)^{p-1} \sum_{i+j=p-1} \frac{(p-1)!}{i!j!} (-1)^i (x_3x_4 + x_5x_6)^i (x_3x_4)^j \\
&= (x_5x_6)^{p-1} \sum_{\substack{i+j=p \\ i,j \neq p}} \frac{(p-1)!}{i!j!} (-1)^i (x_3x_4)^{i-1} (x_3x_4)^j + (x_5x_6)^{p-1} \sum_{i+j=p-1} \frac{(p-1)!}{i!j!} (-1)^i (x_3x_4)^{i+j} \\
&= (x_3x_4x_5x_6)^{p-1} \underbrace{\sum_{\substack{k+j=p-1 \\ k \neq p-1}} \frac{(p-1)!}{k!j!} (-1)^k}_{=(1-1)^{p-1}-1=-1} + (x_3x_4x_5x_6)^{p-1} \underbrace{\sum_{i+j=p-1} \frac{(p-1)!}{i!j!} (-1)^i}_{=(1-1)^{p-1}=0} \\
&= -(x_3x_4x_5x_6)^{p-1} \pmod{f_6, x_1^p, \dots, x_6^p}.
\end{aligned}$$

This implies that there exist $g_{i,j} \in k[x_1, \dots, x_6] / (x_1^p, \dots, x_6^p)$, for $0 \leq i, j < p$, such that

$$(x_1x_2 + x_3x_4 + x_5x_6) \left(\sum_{i,j} g_{i,j} \cdot x_1^i x_2^j \right) = (x_3x_4x_5x_6)^{p-1} \pmod{x_1^p, \dots, x_6^p}.$$

By simple manipulation we see that:

$$\begin{aligned}
&(x_1x_2 + x_3x_4 + x_5x_6) \left(\sum_{i,j} g_{i,j} \cdot x_1^i x_2^j \right) = (x_3x_4x_5x_6)^{p-1} \pmod{x_1^p, \dots, x_6^p} \\
&\left(\sum_{i,j} g_{i,j} \cdot x_1^{i+1} x_2^{j+1} \right) + \left(\sum_{i,j} g_{i,j} (x_3x_4 + x_5x_6) \cdot x_1^i x_2^j \right) = (x_3x_4x_5x_6)^{p-1} \pmod{x_1^p, \dots, x_6^p} \\
&\left(\sum_{i,j} (g_{i-1,j-1} + g_{i,j} (x_3x_4 + x_5x_6)) \cdot x_1^i x_2^j \right) = (x_3x_4x_5x_6)^{p-1} \pmod{x_1^p, \dots, x_6^p}.
\end{aligned}$$

Since $k[x_1, \dots, x_6] / (x_1^p, \dots, x_6^p)$ is a free $k[x_3, \dots, x_6] / (x_3^p, \dots, x_6^p)$ -module on generators $x_1^i x_2^j$, for $0 \leq i, j < p$, we may compare coefficients to obtain:

$$\begin{aligned}
g_{0,0}(x_3x_4 + x_5x_6) &= (x_3x_4x_5x_6)^{p-1} \pmod{x_3^p, \dots, x_6^p} \\
g_{i,i} &= -g_{i+1,i+1}(x_3x_4 + x_5x_6) \pmod{x_3^p, \dots, x_6^p}.
\end{aligned}$$

Using the second relation we obtain:

$$\begin{aligned}
(-1)^{p-1}(x_3x_4x_5x_6)^{p-1} &= (-1)^{p-1}g_{0,0}(x_3x_4 + x_5x_6) \\
&= (-1)^{p-2}g_{1,1}(x_3x_4 + x_5x_6)^2 \\
&= \dots \\
&= -g_{p-2,p-2}(x_3x_4 + x_5x_6)^{p-1} \\
&= g_{p-1,p-1}(x_3x_4 + x_5x_6)^p = 0 \pmod{x_3^p, \dots, x_6^p},
\end{aligned}$$

which is a contradiction for $N = 2n$. The case $N = 2n - 1$ can be treated analogously following the lines of the above. The only difference is that x_5^2 plays a role of x_5x_6 in the above. \square

Remark 6.1.2. We have implemented a `Macaulay2` procedure checking whether a give affine scheme is $W_2(k)$ -liftable. The source code is presented in §8.1.

6.2 Rational, simply connected examples of non-liftable schemes

As described in §5.4 there exist smooth varieties defined over a perfect field k of characteristic p which do not admit a $W_2(k)$ -lifting. In this section, we provide some new examples which apart from being non-liftable mod p^2 avoid other characteristic p pathologies. In particular, they satisfy the following

Good properties

1. they are smooth, projective, rational, and algebraically simply connected,
2. their ℓ -adic integral cohomology rings are generated by algebraic cycles,
3. their integral crystalline cohomology groups are torsion-free F -crystals,
4. they are ordinary in the sense of Bloch–Kato, and of Hodge–Witt type, and their Hodge–de Rham spectral sequence degenerates (cf. §4.2 and §5.2.1 for the relevant definitions).

In contrast, the classical examples often behave pathologically with this respect. For instance, the examples of Raynaud [Ray78] are surfaces of general type, those of Hirokado and Schröer are not of Hodge–Witt type (see [Eke04, Section 4]).

6.2.1 Blow-up of the self-product of a homogeneous space in the graph of the Frobenius morphism

We precede the construction with a few necessary remarks concerning étale sites and ℓ -adic cohomology (see [Mil80, Gro77] for more details).

Étale sites and étale fundamental groups

Definition 6.2.1 (Étale homeomorphism). We say that a morphism $X \rightarrow Y$ is an *étale homeomorphism* if it induces an equivalence on the associated étale sites.

Example 6.2.2. For any scheme X over \mathbb{F}_p the absolute Frobenius morphism $F: X \rightarrow X$ is an étale homeomorphism. Indeed, let $X' \rightarrow X$ be an étale morphism. Then by [Gro77, XIV=XV §1 n° 2, Proposition. 2(c)] the diagram:

$$\begin{array}{ccc} X' & \xrightarrow{F} & X' \\ \downarrow & & \downarrow \\ X & \xrightarrow{F} & X \end{array}$$

is cartesian.

Proposition 6.2.3. *Let X and Y be étale homeomorphic schemes. Then their étale fundamental groups and ℓ -adic cohomology groups are isomorphic.*

It turns out that étale fundamental groups are also invariants of birational proper morphisms.

Theorem 6.2.4 ([Gro71, Exposé X, Corollaire 3.4]). *Let $f: X \rightarrow Y$ be a birational proper morphism of varieties with X normal and Y non-singular. Then $f_*: \pi_1^{\text{ét}}(X) \rightarrow \pi_1^{\text{ét}}(Y)$ is an isomorphism.*

Construction

We now proceed to the construction. Let k be a perfect field of characteristic p . We fix a semi-simple algebraic group G over \mathbb{F}_p , a parabolic subgroup $P \subseteq G$ (reduced), and set $Y = (G/P) \times_{\text{Spec } \mathbb{F}_p} \text{Spec } k$. We assume that the pair (G, P) satisfies the assumption of Theorem 5.3.8, and therefore the k -linear Frobenius morphism

$$F_Y = F \times_{\text{Spec } \mathbb{F}_p} \text{Spec } k: Y \rightarrow Y$$

does not lift to $W_2(k)$. For example, using Remark 5.3.9, we see that Y could be the Grassmannian $\text{Gr}(n, k)$ ($1 < k < n - 1$) or the full flag variety SL_n/B ($n \geq 3$, $B =$ upper-triangular matrices), or a smooth quadric hypersurface in \mathbb{P}^n , $n \geq 4$.

Theorem 6.2.5. *Let $\Gamma_{F_Y} \subseteq Y \times Y$ be the graph of the k -linear Frobenius morphism $F_Y: Y \rightarrow Y$. Let $X = \text{Bl}_{\Gamma_{F_Y}}(Y \times Y)$ be the blow-up of $Y \times Y$ along Γ_{F_Y} , and $X' = \text{Bl}_{\Delta_Y}(Y \times Y)$ the blow-up of $Y \times Y$ along the diagonal. Then X and X' share the good properties listed above and moreover:*

- a) *they are ‘étale homeomorphic,’ i.e., their étale sites are equivalent,*
- b) *their ℓ -adic integral cohomology rings are isomorphic as Galois representation,*
- c) *their integral crystalline cohomology groups are isomorphic torsion-free F -crystals.*

However, X' admits a projective lift to $W(k)$, while X lifts neither to characteristic zero (even formally), nor to $W_2(k)$.

Proof. Good property (1) follows from Bruhat decomposition (see [Spr98, Section 8.3]) and the birational invariance of the étale fundamental group of smooth varieties (cf. Theorem 6.2.4). Properties (2)–(4) follow from Corollary 4.1.4 and Lemma 4.2.5. Property (a) follows from the existence of the cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ f \downarrow & & \downarrow f' \\ Y \times Y & \xrightarrow{\text{id} \times F_Y} & Y \times Y, \end{array}$$

where f and f' are the respective blow-up maps. Indeed, the Frobenius map $F_{X'}: X' \rightarrow X'$ and the composition $(F_Y \times \text{id}) \circ f': X' \rightarrow Y \times Y$ yield a map $v: X' \rightarrow X$ making the diagram

$$\begin{array}{ccccccc} & & F_{X'} & & F_X & & \\ & \curvearrowright & & \curvearrowright & & \curvearrowright & \\ X' & \xrightarrow{v} & X & \xrightarrow{u} & X' & \xrightarrow{v} & X \\ f' \downarrow & & f \downarrow & & f' \downarrow & & f \downarrow \\ Y \times Y & \xrightarrow{F_Y \times \text{id}} & Y \times Y & \xrightarrow{\text{id} \times F_Y} & Y \times Y & \xrightarrow{F_Y \times \text{id}} & Y \times Y \\ & \curvearrowleft & & \curvearrowleft & & \curvearrowleft & \\ & & F_{Y \times Y} & & F_{Y \times Y} & & \end{array}$$

commute. In particular, $v \circ u = F_X$ and $u \circ v = F_{X'}$. Since F_X and $F_{X'}$ are étale homeomorphisms, u and v are étale homeomorphisms as well. Property (b) follows from (a) since the ℓ -adic cohomology depends only on the underlying étale site. Finally, property (c) follows from the blow-up formula given in Corollary 4.1.4. We remark that the crystalline cohomology algebras $H_{\text{cris}}^*(X/W)$ and $H_{\text{cris}}^*(X'/W)$ are not isomorphic, but become so after inverting p .

We now prove that X' admits a projective lifting over $W(k)$ and that X does not lift either to $W_2(k)$ or any ramified extension of $W(k)$. For the first claim, we observe that Y lifts to a projective scheme \mathcal{Y} over $W(k)$ and consequently $\mathcal{X}' = \text{Bl}_{\Delta_{\mathcal{Y}}}(\mathcal{Y} \times_{W(k)} \mathcal{Y})$ is a projective lifting of X' . We now proceed to the second claim. We begin with a proposition addressing Frobenius liftability of homogeneous spaces and describing their cohomological properties necessary to apply deformation theoretic results stated in §3.6.6 and §3.6.2.

Proposition 6.2.6. *Let Y be a homogeneous space over k of a semisimple algebraic group G not isomorphic to any projective space. Then, $H^1(Y, \mathcal{T}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.*

Proof. For the first vanishing, we use [Dem77, Théoreme 2]. The vanishing $H^i(Y, \mathcal{O}_Y) = 0$ is the consequence of Kempf vanishing (i.e., a characteristic p analogue of the Borel–Weil–Bott theorem) as 0 is a dominant weight for the parabolic subgroup of G corresponding to Y . \square

We show X does not lift to $W_2(k)$. Assume the contrary, i.e., that there exists a $W_2(k)$ -lifting of $\text{Bl}_{\Gamma_Y}(Y \times Y)$. By Proposition 3.6.21 there exist two liftings \tilde{Y} and \tilde{Y}' of Y together with a lifting $\tilde{F}: \tilde{Y} \rightarrow \tilde{Y}'$ of $F_Y: Y \rightarrow Y$. However, since $H^1(Y, \mathcal{T}_Y) = 0$, the homogeneous space Y is rigid, which implies that the lifting \tilde{Y}' is isomorphic to \tilde{Y} . This implies that Y is $W_2(k)$ -liftable compatibly with Frobenius, which contradicts Theorem 5.3.8.

Finally, we address non-liftability of X to characteristic zero. Let us assume, for the sake of contradiction, that X lifts to characteristic zero. Any such lifting induces a formal lifting which by Lemma 3.6.21 and rigidity of Y gives a formal lifting of a non-trivial endomorphism $F_Y: Y \rightarrow Y$. By the Grothendieck algebraization theorem the formal lifting of the finite morphism F_Y extends to an algebraic lifting which contradicts Theorem 5.3.10. \square

6.2.2 Examples arising from line configurations

Let k be an algebraically closed field of characteristic p . Let $P = \mathbb{P}^3(\mathbb{F}_p) \subseteq \mathbb{P}_k^3$ be the set of all \mathbb{F}_p -rational points, and let $Y = \text{Bl}_P \mathbb{P}_k^3$, and let L be the set of all lines in \mathbb{P}_k^3 meeting P at least twice. Finally, let $\tilde{L} \subseteq Y$ be the set of the strict transforms of all elements of L , and let $X = \text{Bl}_{\tilde{L}} Y$.

Theorem 6.2.7. *The threefold X has the properties (1)–(4) given at beginning of this section, but it does not admit a lift to any ring A with $pA \neq 0$.*

For the properties (1)–(4), we argue exactly as in the previous section. The proof that X does not deform to any algebra A with $pA \neq 0$ consists of the following three propositions.

Proposition 6.2.8. *Let A be an object of $\mathbf{Art}_{W(k)}(k)$, and suppose that X lifts to A . Then \mathbb{P}_k^3 lifts to A together with all \mathbb{F}_p -rational points and lines, preserving the incidence relations.*

Proof. Let E be the set consisting of the preimages in Y of the elements of P , F the set of preimages in X of the elements of \tilde{L} . Finally, let $Q = (\bigcup \tilde{L}) \cap (\bigcup E)$ (treated as a set of closed points). We have the following chain of natural transformations between various deformation functors:

$$\mathrm{Def}_X \xleftarrow{\simeq} \mathrm{Def}_{X,F} \longrightarrow \mathrm{Def}_{Y,\tilde{L}} \xleftarrow{\simeq} \mathrm{Def}_{Y,\tilde{L} \cup E} \xleftarrow{\simeq} \mathrm{Def}_{Y,\tilde{L} \cup E \cup Q} \longrightarrow \mathrm{Def}_{\mathbb{P}_k^3, L \cup P}.$$

We remind the reader of our convention (cf. Remark 3.6.8) that for a family of closed subschemes $Z = \{Z_i\}_{i \in I}$ of a scheme X indexed by a preorder I , $\mathrm{Def}_{X,Z}$ is the functor of deformations of X , together with embedded deformations of Z_i , preserving the inclusion relations $Z_i \subseteq Z_{i'}$ for $i \leq i'$. Above, we give the families $F, \tilde{L}, \tilde{L} \cup E$ the trivial order, and order $\tilde{L} \cup E \cup Q$ and $L \cup P$ by inclusion. In particular, the functor $\mathrm{Def}_{Y,\tilde{L} \cup E \cup Q}$ parametrises deformations of Y together with the strict transforms of the \mathbb{F}_p -rational lines (i.e., \tilde{L}) and the preimages of the \mathbb{F}_p -rational points (i.e., E) in \mathbb{P}_k^3 such that their mutual intersections are flat over the base (i.e., induce a compatible embedded deformation of Q). Similarly, $\mathrm{Def}_{\mathbb{P}_k^3, L \cup P}$ is the functor of deformations of \mathbb{P}_k^3 together with all the \mathbb{F}_p -rational points and lines, preserving the incidence relations. We discuss the maps in this chain below.

The maps $\mathrm{Def}_{X,F} \rightarrow \mathrm{Def}_X$, $\mathrm{Def}_{Y,\tilde{L} \cup E} \rightarrow \mathrm{Def}_{Y,\tilde{L}}$, and $\mathrm{Def}_{Y,\tilde{L} \cup E \cup Q} \rightarrow \mathrm{Def}_{Y,\tilde{L} \cup E}$ are the forgetful transformations. The first two are isomorphisms by Proposition 3.6.9 (2), and the last map is an isomorphism by Corollary 2.1.7 applied to the local equations of E and \tilde{L} .

The maps $\mathrm{Def}_{X,F} \rightarrow \mathrm{Def}_{Y,\tilde{L}}$ and $\mathrm{Def}_{Y,\tilde{L} \cup E \cup Q} \rightarrow \mathrm{Def}_{\mathbb{P}_k^3, L \cup P}$ are the maps of Proposition 3.6.9 (1). For the latter, strictly speaking, Proposition 3.6.9 (1) yields a map $\mathrm{Def}_{Y,\tilde{L} \cup E \cup Q} \rightarrow \mathrm{Def}_{\mathbb{P}_k^3, Z}$, where $Z = \{Z_s\}_{s \in S}$ is the ‘image’ of $\tilde{L} \cup E \cup Q$, defined as follows. Let $S = L \sqcup P \sqcup K$ where $K = \{(x, \ell) \in P \times L : x \in \ell\}$, given the ordering whose nontrivial relations are $(\ell, x) \leq \ell$ and $(\ell, x) \leq x$ for $x \in P, \ell \in L, (x, \ell) \in K$. Then set $Z_\ell = \ell$ for $\ell \in L$, $Z_x = x$ for $x \in P$, and $Z_{(x, \ell)} = x$ for $(x, \ell) \in K$. For an algebra A , an element of $\mathrm{Def}_{\mathbb{P}_k^3, Z}$ is thus given by a deformation of \mathbb{P}_k^3 together with deformations of the $\ell \in L$, $x \in X$, and $Z_{(x, \ell)} = x$ for $(x, \ell) \in K$, preserving the relations $\tilde{Z}_{(x, \ell)} \subseteq \tilde{x}$ and $\tilde{Z}_{(x, \ell)} \subseteq \tilde{\ell}$ for $(x, \ell) \in K$ (here the tildes mean the corresponding deformations over A). But each x is a point, so $\tilde{Z}_{(x, \ell)} \subseteq \tilde{x}$ implies $\tilde{Z}_{(x, \ell)} = \tilde{x}$, and the deformation of (\mathbb{P}_k^3, Z) simplifies to a deformation of $(\mathbb{P}_k^3, L \cup P)$ preserving the incidence relations. Thus $\mathrm{Def}_{\mathbb{P}_k^3, Z}$ can be identified with $\mathrm{Def}_{\mathbb{P}_k^3, L \cup P}$. \square

Remark 6.2.9. Since we will have to deal with a little bit of elementary projective geometry and matroid representability over arbitrary rings, let us introduce some definitions and fix some conventions.

Let A be a local ring with residue field k . A projective n -space \mathbb{P} over A is an A -scheme of the form $\mathbb{P}_A(V)$ for a finitely generated free A -module V , and a d -dimensional linear subspace L of \mathbb{P} is the image of a map $\mathbb{P}_A(W) \rightarrow \mathbb{P}_A(V)$ induced by a surjective map $V \rightarrow W$. Zero-dimensional linear subspaces of \mathbb{P} can be identified with the set $\mathbb{P}(A)$. If $x, y \in \mathbb{P}(A)$ are points, given by the surjections $V \rightarrow L_x$ and $V \rightarrow L_y$ respectively, whose images in $\mathbb{P}(k)$ are distinct, there exists a unique line (i.e., a one-dimensional linear subspace) $\ell(x, y)$ containing both x and y induced by the surjection $V \rightarrow L_x \oplus L_y$.

We say that points x, y, z are collinear (resp. coplanar) if they lie on one line (resp. 2-dimensional subspace). If x_0, \dots, x_n, z are points whose images in $\mathbb{P}(k)$ are in general position, there exists a unique isomorphism $\phi : \mathbb{P} \rightarrow \mathbb{P}_A^n$ such that $\phi(x_i) = e_i := (0 : \dots : 0 : 1 : 0 : \dots : 0)$ (with 1 on the i -th coordinate) and $\phi(z) = f := (1 : \dots : 1)$. In particular, if $A \in \mathbf{Art}_{W(k)}(k)$, and S is a configuration of linear subspaces of \mathbb{P}_k^n ordered by inclusion, containing the points $e'_i = (0 : \dots : 0 : 1 : 0 : \dots : 0)$ and $f' = (1 : \dots : 1)$, we can identify the deformation functor $\mathrm{Def}_{\mathbb{P}_k^n, S}^{\mathbb{P}_k^n}(A)$ with the set of all families of linear subspaces \tilde{S} in \mathbb{P}_A^n which yield the given S upon restriction to k , and such that $\tilde{e}'_i = e_i$ and $\tilde{f}' = f$.

Proposition 6.2.10. *Suppose that \mathbb{P}_k^3 lifts to an Artinian $W(k)$ -algebra A together with all \mathbb{F}_p -rational points, preserving collinearity. Then \mathbb{P}_k^2 lifts to A together with all \mathbb{F}_p -rational points, preserving collinearity.*

Proof. The key observation is that coplanarity is also preserved, i.e., that $\mathrm{Def}_{\mathbb{P}_k^3, L \cup P}^{\mathbb{P}_k^3} = \mathrm{Def}_{\mathbb{P}_k^3, H \cup L \cup P}^{\mathbb{P}_k^3}$, where H denotes the set of all \mathbb{F}_p -rational hyperplanes in \mathbb{P}_k^3 (with $H \cup L \cup P$ ordered by inclusion). Indeed, let A be an object of $\mathbf{Art}_{W(k)}(k)$, and suppose we are given an element of $\mathrm{Def}_{\mathbb{P}_k^3, L \cup P}^{\mathbb{P}_k^3}(A)$, which by simple rigidification (e.g., the requirement that the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, and $(1 : 1 : 1)$ do not deform) can be identified with a configuration of points \tilde{x} and lines $\tilde{\ell}$ in \mathbb{P}_A^3 , indexed by P and L respectively, such that $\tilde{x} \subseteq \tilde{\ell}$ whenever $x \in \ell$. To get an element of $\mathrm{Def}_{\mathbb{P}_k^3, H \cup L \cup P}^{\mathbb{P}_k^3}$, it suffices to show that whenever $x_1, x_2, x_3, x_4 \in P$ is a quadruple of coplanar points, the points $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{P}^3(A)$ are coplanar. If two of the points x_i coincide, there is nothing to show, and similarly if all four lie on a line. Otherwise, let $\ell_{12} = \ell(x_1, x_2)$ and $\ell_{34} = \ell(x_3, x_4)$, then $\tilde{\ell}_{12} = \ell(\tilde{x}_1, \tilde{x}_2)$ and $\tilde{\ell}_{34} = \ell(\tilde{x}_3, \tilde{x}_4)$. Since the x_i are coplanar, the lines ℓ_{12} and ℓ_{34} intersect in a unique point $y \in P$. Then $\tilde{y} \in \tilde{\ell}_{12} \cap \tilde{\ell}_{34} = \ell(\tilde{x}_1, \tilde{x}_2) \cap \ell(\tilde{x}_3, \tilde{x}_4)$. Thus the hyperplane through $\tilde{y}, \tilde{x}_1, \tilde{x}_2$ yields a lift of the hyperplane through x_1, x_2, x_3, x_4 .

Since coplanarity is preserved, we can forget everything except for the plane $x_0 = 0$ (say) and get a desired lifting of \mathbb{P}_k^2 . Equivalently, we could have used a projection from one of the \mathbb{F}_p -rational points. \square

To finish, we prove that the matroid $\mathbb{P}^2(\mathbb{F}_p)$ does not admit a projective representation over any ring A with $pA \neq 0$, that is, there exist no function $\rho : \mathbb{P}^2(\mathbb{F}_p) \rightarrow \mathbb{P}^2(A)$ taking triples of collinear points to triples of collinear points. If A is a field, this is well-known (cf. e.g. [Gor88, Theorem 2]). We check that the proof works for arbitrary rings.

Proposition 6.2.11. *Let A be a ring, $\rho : \mathbb{P}^2(\mathbb{F}_p) \rightarrow \mathbb{P}^2(A)$ a map taking triples of collinear points to triples of collinear points. Then $pA = 0$.*

Proof. Changing coordinates in $\mathbb{P}^2(A)$, we can assume that

$$\rho(1 : 0 : 0) = (1 : 0 : 0), \quad \rho(0 : 1 : 0) = (0 : 1 : 0), \quad \rho(0 : 0 : 1) = (0 : 0 : 1),$$

and $\rho(1 : 1 : 1) = (1 : 1 : 1)$. Thus

$$\rho(1 : 1 : 0) = \rho(\ell((0 : 0 : 1), (1 : 1 : 1)) \cap \ell((1 : 0 : 0), (0 : 1 : 0))) = (1 : 1 : 0)$$

as well. For $n \in \mathbb{Z}$, let $P_n = (n : 0 : 1)$, $Q_n = (n + 1 : 1 : 1) \in \mathbb{P}^2(\mathbb{F}_p)$, and let $P'_n, Q'_n \in \mathbb{P}^2(A)$ be the points with the same coordinates as P_n, Q_n . We check

by induction on $n \geq 0$ that $\rho(P_n) = P'_n$ and $\rho(Q_n) = Q'_n$: the base case is ok, and for the induction step we note that $P_n = \ell(Q_{n-1}, (0 : 1 : 0)) \cap \ell(P_0, (1 : 0 : 0))$, $Q_n = \ell(P_n, (1 : 1 : 0)) \cap \ell(Q_0, (1 : 0 : 0))$, and that the same statements hold with the primes (see Figure 6.2.2). Thus $(p : 0 : 1) = \rho(p : 0 : 1) = \rho(0 : 0 : 1) = (0 : 0 : 1)$, and hence $p = 0$ in A . \square

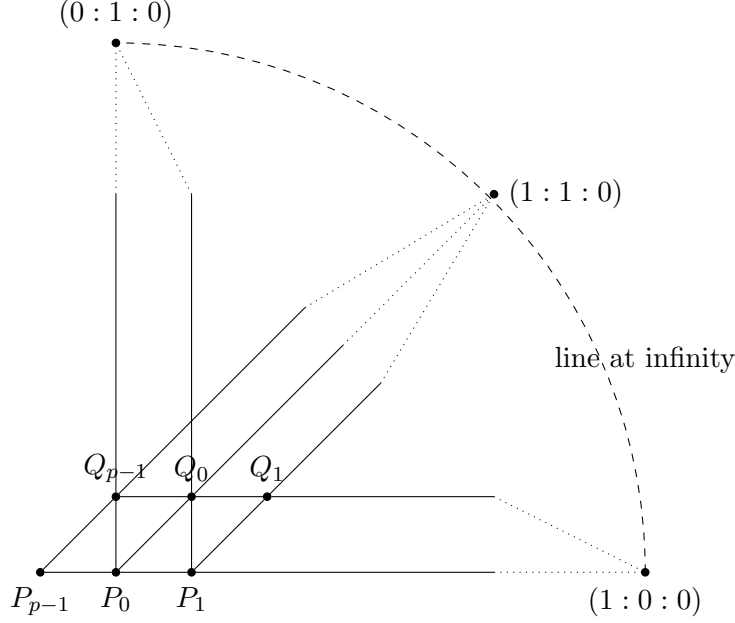


Figure 6.1: *Proof of Proposition 6.2.11*

Remark 6.2.12. Note that the proof exhibits a sub-matroid (denoted M_p in [Gor88]) consisting of $2p + 3$ points sharing the desired property of $\mathbb{P}^2(\mathbb{F}_p)$. This means that in our second non-liftable example we could have blown up a smaller configuration of $2p + 4$ points and (strict transforms of) $4p + 7$ lines between them.

Remark 6.2.13. With the same proof, one can construct similar examples in higher dimensions: blow up \mathbb{P}_k^n ($n \geq 3$) in all \mathbb{F}_p -rational points, (strict transforms of) lines, planes, and so on. Such varieties were studied in [RTW13, Definition 1.2] in relation to automorphisms of the Drinfeld half-space.

Remark 6.2.14. We also remark that in [Lan16, Proposition 8.4] it is proved that a pair (X, D) where X is the blow-up of \mathbb{P}_k^2 in all \mathbb{F}_p -rational points and D is a union of strict transforms of at least $4p - 3$ \mathbb{F}_p -rational lines does not lift to $W_2(k)$. The argument above proves that the matroid M_p leads to a non-liftable example with a fewer number of lines equal to $2p + 3$. We do not know whether $2p + 3$ is the minimal number of lines necessary to exhibit $W_2(k)$ non-liftability.

Remark 6.2.15. Related non-liftable examples also appear in [BHH87, Section 3.5J] and [Eas08]. The constructions are based on ramified coverings along purely characterstic p lines configurations instead of blow-ups.

6.3 Explicit $W_2(k)$ -lifting of a Frobenius-split scheme

We now reprove the classical result stating that any Frobenius split scheme X over a perfect field k of characteristic p is $W_2(k)$ -liftable (see Proposition 5.1.5). We give an explicit construction which depends functorially on the Frobenius splitting $\varphi : F_{X*}\mathcal{O}_X \rightarrow \mathcal{O}_X$, in fact as a certain subscheme of the second Witt scheme $(X, W_2(\mathcal{O}_X))$.

6.3.1 Functorial setting

By *Frobenius split scheme* we mean a pair (X, φ_X) consisting of a scheme over k and a Frobenius splitting $\varphi_X : F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$. Moreover, a *morphism of Frobenius split schemes* (X, φ_X) and (Y, φ_Y) is a morphism of schemes $\pi : X \rightarrow Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{\pi^\#} & \pi_*\mathcal{O}_X \\ \varphi_Y \uparrow & & \uparrow \pi_*\varphi_X \\ & F^\# & \downarrow \pi_*F^\# \\ F_*\mathcal{O}_Y & \xrightarrow{F_*\pi^\#} & F_*\pi_*\mathcal{O}_X = \pi_*F_*\mathcal{O}_X. \end{array}$$

This is equivalent to the relation $\pi^\# \circ \varphi_Y = (F_*\pi^\#) \circ (\pi_*\varphi_X)$. In case of an affine morphism corresponding to a homomorphism $f : A \rightarrow B$ this is equivalent to the equality $f\varphi_A = \varphi_B f$. The notions described above allow us to introduce the category of Frobenius split schemes.

Definition 6.3.1. We define a category $\text{Sch}_k^{\text{split}}$ of Frobenius split schemes over k as a category with the set of objects consisting of all Frobenius split schemes (X, ϕ_X) and with the set of morphisms consisting of morphisms of Frobenius split schemes $f : (X, \phi_X) \rightarrow (Y, \phi_Y)$.

6.3.2 The construction of $W_2^\varphi(A)$

First we give a construction in the affine case. A simple consequence of Corollary 2.1.10 is that a functorial construction $A \mapsto W_2(A)$ does not give a flat lifting (the ideal $pW_2(A)$ is not equal to $\text{Ker}[W_2(A) \rightarrow A]$). Therefore we proceed in a different manner taking the Frobenius splitting into account. Let (A, φ) be a k -algebra together with a Frobenius splitting φ . We claim that a flat lifting of $\text{Spec}(A)$ over $W_2(k)$ is given by the following construction which is functorial with respect to natural mappings of \mathbb{F}_p rings with Frobenius splitting, that is, ring homomorphisms commuting with p^{-1} -linear splitting operators. The Witt vector operators $+_W$ and \cdot_W used below are described in Definition 2.1.1.

Definition 6.3.2. The ring of twisted Witt vectors $W_2^\varphi(A)$ associated to a characteristic p ring with a splitting (A, φ) is the set $A \times A$ with addition and multiplication given by the following formulas:

$$(a_0, a_1) +_\varphi (b_0, b_1) \stackrel{\text{def}}{=} (\text{id} \times \varphi)[(a_0, a_1^p) +_W (b_0, b_1^p)] = (a_0 + b_0, a_1 + b_1 - (\varphi \circ P)(a_0, b_0)) \quad (6.2)$$

$$(a_0, a_1) \cdot_\varphi (b_0, b_1) \stackrel{\text{def}}{=} (\text{id} \times \varphi)[(a_0, a_1^p) \cdot_W (b_0, b_1^p)] = (a_0 b_0, a_0 b_1 + b_0 a_1). \quad (6.3)$$

This formulas can be also described by the following diagrams:

$$\begin{array}{ccc}
(A \times A)^{\times 2} & \xrightarrow{+\varphi} & A \times A \\
\downarrow (\text{id} \times F)^{\times 2} & & \uparrow \text{id} \times \varphi \\
(A \times A)^{\times 2} & \xrightarrow{+W} & A \times A
\end{array}
\qquad
\begin{array}{ccc}
(A \times A)^{\times 2} & \xrightarrow{\cdot \varphi} & A \times A \\
\downarrow (\text{id} \times F)^{\times 2} & & \uparrow \text{id} \times \varphi \\
(A \times A)^{\times 2} & \xrightarrow{\cdot W} & A \times A.
\end{array}$$

The verification that the above definition gives indeed a structure of an associative and commutative ring is a straightforward computation. For the purpose of illustration we prove associativity:

$$\begin{aligned}
(a_0, a_1) \cdot_{\varphi} ((b_0, b_1) +_{\varphi} (c_0, c_1)) &= (a_0, a_1) \cdot_{\varphi} (b_0 + c_0, b_1 + c_1 - (\varphi \circ P)(b_0, c_0)) \\
&= (a_0 b_0 + a_0 c_0, a_0(b_1 + c_1 - (\varphi \circ P)(b_0, c_0)) + (b_0 + c_0)a_1) \\
&= (a_0 b_0 + a_0 c_0, a_0 b_1 + a_1 b_0 + a_0 c_1 + a_1 c_0 - a_0(\varphi \circ P)(b_0, c_0)) \\
&= (a_0 b_0 + a_0 c_0, a_0 b_1 + a_1 b_0 + a_0 c_1 + a_1 c_0 - (\varphi \circ P)(a_0 b_0, a_0 c_0)) \\
&= (a_0 b_0, a_0 b_1 + b_0 a_1) +_{\varphi} (a_0 c_0, a_0 c_1 + c_0 a_1) \\
&= (a_0, a_1) \cdot_{\varphi} (b_0, b_1) +_{\varphi} (a_0, a_1) \cdot_{\varphi} (c_0, c_1).
\end{aligned}$$

The middle transition follows from the relation (note that ϕ is p^{-1} -linear):

$$a(\varphi \circ P)(b, c) = \varphi(a^p P(b, c)) = (\varphi \circ P)(ab, ac).$$

We are now ready to prove:

Theorem 6.3.3. *Any morphism of Frobenius-split algebras $f : (A, \varphi_A) \rightarrow (B, \varphi_B)$ induces a morphism $W_2^{\varphi}(f) : W_2^{\varphi_A}(A) \rightarrow W_2^{\varphi_B}(B)$. Moreover, the ring $W_2^{\varphi_A}(A)$ is a flat $W_2(k)$ -lifting of A .*

Proof. Firstly, we observe that our construction is functorial. For this purpose, we define $W_2^{\varphi}(f) : W_2^{\varphi_A}(A) \rightarrow W_2^{\varphi_B}(B)$ by the following diagram where the horizontal mappings are given by the natural identifications of $W_2^{\varphi_A}(A)$ and $A \times A$.

$$\begin{array}{ccc}
W_2^{\varphi_A}(A) & \xrightarrow{W_2^{\varphi}(f)} & W_2^{\varphi_B}(B) \\
\downarrow & & \uparrow \\
A \times A & \xrightarrow{f \times f} & B \times B
\end{array}$$

The compatibility of this definition with addition and multiplication follows directly from the formulas and the condition $f \circ \varphi_A = \varphi_B \circ f$.

Consequently, we show that $W_2^{\varphi}(A)$ is flat over $W_2(k)$. Clearly the projection onto the first factor gives a surjective ring homomorphism from $W_2^{\varphi}(A)$ to A . The kernel I of this homomorphism is given by the ideal of elements of the form $(0, a_1) \in W_2^{\varphi}(A)$ for $a_1 \in A$, and is therefore generated by a single element $t = (0, 1)$. In case of a perfect field k with a Frobenius splitting $\gamma_k(\alpha) = \sqrt[p]{\alpha}$ we obtain a diagram:

$$\begin{array}{ccc}
(k \times k)^{\times 2} & \xrightarrow{+\varphi} & k \times k \\
\downarrow (\text{id} \times F)^{\times 2} & & \uparrow \text{id} \times \varphi \\
(k \times k)^{\times 2} & \xrightarrow{+W} & A \times A
\end{array}
\qquad
\begin{array}{ccc}
(k \times k)^{\times 2} & \xrightarrow{\cdot \varphi} & k \times k \\
\downarrow (\text{id} \times F)^{\times 2} & & \uparrow \text{id} \times \varphi \\
(k \times k)^{\times 2} & \xrightarrow{\cdot W} & k \times k.
\end{array}$$

Consequently, the bijective mapping $\text{id} \times F$ gives an isomorphism between the rings $W_2^{\gamma_k}(k)$ and $W_2(k)$. This means that natural homomorphism $W_2(k) \simeq W_2^{\gamma_k}(k) \rightarrow W_2^{\varphi}(A)$ induced by the embedding $k \rightarrow A$ sends $p = (0, 1) \in W_2(k)$ to $(0, 1) \in W_2^{\varphi}(A)$ and therefore $t = p$. This allows us to apply Corollary 2.1.10 to conclude that $W_2^{\varphi}(A)$ is indeed a flat lifting of A . \square

Glueing to a twisted Witt scheme

In order to globalise the above construction we shall show that it behaves well with respect to localisation. For this purpose, we prove the following lemma resembling the computation given in [Blo77, Lemma 5.1].

Lemma 6.3.4. *Let A be a k -algebra. For any lifting $(f, c) \in W_2^\varphi(A)$ of $f \in A$ the natural homomorphism $i_f : W_2^\varphi(A)_{(f, c)} \rightarrow W_2^\varphi(A_f)$ is an isomorphism.*

Before proceeding to the proof, we present a series of useful equalities in $W_2^\varphi(A)$ which follows directly from addition and multiplication formulas:

$$(f, c)^{-1} = (1/f, -c/f^2) \quad (6.4)$$

$$(a, b)^n = (a^n, na^{n-1}b). \quad (6.5)$$

Proof of Lemma 6.3.4. It is sufficient to show that i_f is bijective. To prove it, we first apply equalities (6.4) and (6.5) to obtain:

$$i_f \left(\frac{(a, b)}{(f, c)^n} \right) = (a, b) \cdot_\varphi \left(\frac{1}{f}, \frac{-c}{f^2} \right)^n = (a, b) \cdot_\varphi \left(\frac{1}{f^n}, \frac{-nc}{f^{n+1}} \right) = \left(\frac{a}{f^n}, \frac{fb - nac}{f^{n+1}} \right).$$

Now we perform a straightforward inspection using the following two facts: (i) $\frac{a}{f^n} = 0$ if and only if $f^s a = 0$ for some $s \in \mathbb{N}$, (ii) the mapping $A_f \ni d \mapsto d - \frac{nac}{f^{n+1}}$ is bijective for any choice of parameters (n, a, c) . \square

We are now ready to prove the theorem. Let $\text{Sch}_{W_2(k)}^{\text{flat}}$ be the category of flat schemes over the ring of second Witt vectors.

Theorem 6.3.5. *There exists a functor W_2^φ from the category $\text{Sch}_k^{\text{split}}$ to the category $\text{Sch}_{W_2(k)}^{\text{flat}}$ such that the following diagram of functors:*

$$\begin{array}{ccc} \text{Sch}_k^{\text{split}} & \xrightarrow{W_2^\varphi} & \text{Sch}_{W_2(k)}^{\text{flat}} \\ & \searrow \text{forget} & \swarrow \otimes k \\ & \text{Sch}_k & \end{array}$$

is commutative, that is, the scheme $W_2^\varphi((X, \varphi_X))$ is a functorial flat $W_2(k)$ -lifting of X .

Proof. In case of affine schemes $(\text{Spec}(A), \phi)$ it is sufficient to use the construction of $W_2^\phi(A)$. In general, we proceed as follows. For a scheme (X, \mathcal{O}_X) we consider a ringed space $(X, W_2^\varphi(\mathcal{O}_X))$ where $W_2^\varphi(\mathcal{O}_X)$ is a sheaf equal to $\mathcal{O}_X \times \mathcal{O}_X$ set theoretically and with the ring structure defined by the assignment $U \mapsto W_2^\varphi(\mathcal{O}_X(U))$. The sheaf transition maps are induced by the functoriality of the construction given in Theorem 6.3.3. Consequently, Lemma 6.3.4 proves that locally over the affine subset $V = \text{Spec}(A)$ of (X, \mathcal{O}_X) , the sheaf $W_2^\varphi(\mathcal{O}_X)|_V$ is isomorphic to $W_2^\varphi(\widetilde{\mathcal{O}_X(V)})$ and therefore $(X, W_2^\varphi(\mathcal{O}_X))$ is in fact a scheme over $W_2(k)$. \square

Remark 6.3.6. Since ordinary Calabi-Yau varieties are uniquely Frobenius split (cf. Proposition 4.4.8), our construction leads to an explicit and functorial $W_2(k)$ -lifting of such varieties.

6.3.3 Alternative view on the construction

As suggested by Piotr Achinger, the above construction might be alternatively described as a certain subscheme of the second Witt scheme $(X, W_2(\mathcal{O}_X))$. The construction goes as follows.

Firstly, we observe that $W_2(\mathcal{O}_X)$ is an ring extension of \mathcal{O}_X by the square-zero ideal $F_{X*}\mathcal{O}_X$. In particular, the $W_2(\mathcal{O}_X)$ -module structure of $F_{X*}\mathcal{O}_X$ factors through the natural surjection $W_2(\mathcal{O}_X) \rightarrow \mathcal{O}_X$. The splitting φ gives rise to a submodule of $F_{X*}\mathcal{O}_X$ given by $\text{Ker}(\varphi) \simeq F_{X*}\mathcal{O}_X/\mathcal{O}_X$ and consequently a subideal of $F_{X*}\mathcal{O}_X \triangleleft W_2(\mathcal{O}_X)$. We define $W_2^{\phi, alt}(\mathcal{O}_X)$ to be the quotient sheaf $W_2(\mathcal{O}_X)/\text{Ker}(\varphi)$. We now prove that $W_2^{\phi, alt}(\mathcal{O}_X)$ is in fact the same scheme as $W_2^\varphi((X, \mathcal{O}_X))$.

Proposition 6.3.7. *The constructions $W_2^\varphi(\mathcal{O}_X)$ and $W_2^{\varphi, alt}(\mathcal{O}_X)$ give rise to isomorphic schemes.*

Proof. Indeed, the construction of $W_2^{\varphi, alt}(\mathcal{O}_X)$ can be described by the following diagram:

$$\begin{array}{ccccccc}
 & & F_{X*}\mathcal{O}_X/\mathcal{O}_X & \xrightarrow{\simeq} & \text{Ker}(\phi) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & F_{X*}\mathcal{O}_X & \longrightarrow & W_2(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\
 & & \downarrow \varphi & & \downarrow c_\varphi & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & W_2^{\varphi, alt}(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X \longrightarrow 0,
 \end{array}$$

where $c_\varphi : W_2(\mathcal{O}_X) \rightarrow W_2^{\varphi, alt}(\mathcal{O}_X)$ is a ring homomorphism. We see that the ring operations in $W_2^{\varphi, alt}(\mathcal{O}_X)$ are given by a set-theoretic lifting to $W_2(\mathcal{O}_X)$ (e.g., $\text{id} \times F$ given in Definition 6.3.2), followed by the corresponding operations in $W_2(\mathcal{O}_X)$ and then the quotient map c_φ . This is exactly the description given by the diagrams in Definition 6.3.2. \square

6.4 $W_2(k)$ -liftability in families

We now proceed to the proof that under suitable conditions Witt vector liftability is a constructible property, i.e., for a locally finite type family $f: X \rightarrow S$ the set of closed points $s \in S$ such that the fibre X_s is $W_2(k(s))$ -liftable is constructible. We begin with a definition encompassing the properties of a morphism necessary to prove constructibility.

Definition 6.4.1. We say that a morphism $f: X \rightarrow S$ is *strongly equicohomological* if it is locally of finite type and flat, and for any $i \in \{0, \dots, \dim S + 2\}$ the higher direct image $R^i f_* R\mathcal{H}om(L_{X/S}, \mathcal{O}_X)$ is a locally free \mathcal{O}_S -module (potentially of infinite rank).

Strongly equicohomological families satisfy the following crucial property motivated by the application for deformation theory.

Lemma 6.4.2. *Let $f: X \rightarrow S$ be a strongly equicohomological morphism with a smooth target S . For any morphism $T \rightarrow S$ the sheaf $R^2 f_* R\mathcal{H}om(L_{X/S}, \mathcal{O}_X)$ satisfies the base change property, i.e., for any cartesian diagram:*

$$\begin{array}{ccc}
 X_T & \xrightarrow{j} & X \\
 \downarrow f_T & & \downarrow f \\
 T & \xrightarrow{i} & S.
 \end{array}$$

there exists a natural isomorphism $i^* R^2 f_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X) \simeq R^2 f_{T*} \mathcal{R}Hom(L_{X_T/T}, \mathcal{O}_{X_T})$.

Proof. By definition strongly equicohomological morphisms are flat and therefore by Lemma 2.3.3 there exists a natural quasi-isomorphism:

$$Li^* Rf_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X) \rightarrow Rf_{T*} Lj^* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X). \quad (6.6)$$

By Lemma 2.3.4 and the base change property of cotangent complex (see Theorem 3.1.5) we obtain an isomorphism:

$$Lj^* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X) \simeq \mathcal{R}Hom(Lj^* L_{X/S}, Lj^* \mathcal{O}_X) \simeq \mathcal{R}Hom(L_{X_T/T}, \mathcal{O}_{X_T}).$$

This implies that the right hand side of (6.6) is isomorphic to $Rf_{T*} \mathcal{R}Hom(L_{X_T/T}, \mathcal{O}_{X_T})$. By taking cohomology, for any $k \in \mathbb{Z}$ we obtain an isomorphism:

$$L^k i^* Rf_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X) \rightarrow R^k f_{T*} \mathcal{R}Hom(L_{X_T/T}, \mathcal{O}_{X_T}).$$

By Lemma 2.3.2 there exists a convergent spectral sequence:

$$E_2^{pq} = L^p i^* \mathcal{H}^q(Rf_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X)) \Rightarrow L^{p+q} i^* Rf_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X).$$

The terms on E_2 page are isomorphic to $L^p i^* R^q f_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X)$ and therefore, since $f: X \rightarrow S$ is strongly equicohomological, $E_2^{pq} = 0$ for $(p, q) \in \mathbb{Z}_{<0} \times \{0, \dots, \dim S + 2\}$ and $p > 0$. Moreover, by the smoothness of S we see that $E_2^{pq} = 0$ for $p < -\dim S$ and consequently $E_r^{-p, p+2} \simeq E_{r+1}^{-p, p+2}$ for any $p \in \mathbb{Z}$ and $r \geq 2$. This means that the natural filtration induced on $L^2 i^* Rf_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X)$ by the spectral sequence consists of a single term $i^* R^2 f_* \mathcal{R}Hom(L_{X/S}, \mathcal{O}_X)$. This finishes the proof. \square

As a corollary we obtain:

Lemma 6.4.3. *Let $f: X \rightarrow S$ be a strongly equicohomological morphism with a smooth affine target. Then, the set of geometric closed points $\bar{s} \in S$ such that $X_{\bar{s}}$ is $W_2(k(\bar{s}))$ -liftable is closed.*

Proof. By the assumptions on S we see that there exists a $W_2(k)$ -lifting \tilde{S} of S . Therefore, there is a relative obstruction class $\sigma_f \in \text{Ext}^2(L_{X/S}, f^*(p\mathcal{O}_{\tilde{S}})) = \text{Ext}^2(L_{X/S}, \mathcal{O}_X)$ (see Theorem 3.2.14) which vanishes if and only if there exists a flat \tilde{S} -scheme \tilde{X} fitting into the cartesian diagram:

$$\begin{array}{ccc} X & \dashrightarrow & \tilde{X} \\ \downarrow f & & \downarrow \tilde{f} \\ S & \longrightarrow & \tilde{S} \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(W_2(k)). \end{array}$$

Since the morphism $\tilde{S} \rightarrow \text{Spec}(W_2(k))$ is formally smooth, any geometric closed point $\bar{s} \in S$ can be lifted to a mapping $\tilde{i}: \text{Spec}(W_2(k(\bar{s}))) \rightarrow \tilde{S}$. We denote by $i_{\bar{s}}: X_{\bar{s}} \rightarrow X$ the closed immersion defined by the diagram (with a cartesian square):

$$\begin{array}{ccccc} & & X_s & & \\ & \swarrow i_s & \downarrow f_s & \searrow & \\ X & & \text{Spec}(k(s)) & \longrightarrow & \text{Spec}(W_2(k(s))) \\ \downarrow f & \swarrow i & & \nwarrow \tilde{i} & \\ S & \longrightarrow & \tilde{S} & & \end{array}$$

and by $\sigma_{\bar{s}} \in \text{Ext}^2(L_{X_{\bar{s}}/\bar{s}}, \mathcal{O}_{X_{\bar{s}}})$ the associated obstruction class to lifting $X_{\bar{s}}/k(\bar{s})$ to $W_2(k(\bar{s}))$. Since, S is affine, the $\Gamma(S, \mathcal{O}_S)$ -module $\text{Ext}^2(L_{X/S}, \mathcal{O}_X)$ is isomorphic to the module of global sections of the sheaf

$$R^2 f_* R\mathcal{H}om(L_{X/S}, \mathcal{O}_X),$$

and therefore by Lemma 6.4.2 we obtain a specialisation isomorphism:

$$\begin{aligned} \psi_s : \text{Ext}^2(L_{X/S}, \mathcal{O}_X) \otimes_{\mathcal{O}_S} k(\bar{s}) &= i^* R^2 f_* R\mathcal{H}om(L_{X/S}, \mathcal{O}_X) \\ &\simeq R^2 f_{\bar{s}*} R\mathcal{H}om(L_{X_{\bar{s}}/k(\bar{s})}, \mathcal{O}_{X_{\bar{s}}}) = \text{Ext}^2(L_{X_{\bar{s}}/k(\bar{s})}, \mathcal{O}_{X_{\bar{s}}}). \end{aligned}$$

By Lemma 3.4.2 we see that obstructions $\sigma_f \in \text{Ext}^2(L_{X/S}, \mathcal{O}_X)$ and $\sigma_s \in \text{Ext}^2(L_{X_s/k(s)}, \mathcal{O}_{X_s})$ fit into commutative diagram:

$$\begin{array}{ccc} Li_s^* L_{X/S} & \xrightarrow{i_s^* \sigma_f} & Li_s^* \mathcal{O}_X \\ \downarrow di_s & & \downarrow \\ L_{X_{\bar{s}}/k(\bar{s})} & \xrightarrow{\sigma_{\bar{s}}} & \mathcal{O}_{X_{\bar{s}}}, \end{array}$$

and therefore the fibre $\psi_{\bar{s}}([\sigma_f])$ of an obstruction class $\sigma_f \in \text{Ext}^2(L_{X/S}, \mathcal{O}_X)$ is equal to the obstruction $\sigma_s \in \text{Ext}^2(L_{X_s/k(s)}, \mathcal{O}_{X_s})$. Note that here we implicitly use the fact that the second part of the isomorphism (6.6) in Lemma 6.4.2 is given by the differential dj . Closedness then follows from the fact that the zero set of a section of a free module is closed. \square

In order to obtain a general statement for morphism which are not strongly equico-homological we need the following lemma preceded with a notational remark. We say that a morphism $f : X \rightarrow S$ can be *stratified into morphisms satisfying property P* if there exists a stratification of the target $\bigcup_i S_i = S$ into locally closed subschemes $S_i \subset S$ such that the base change $f_i : X \times_S S_i \rightarrow S_i$ satisfies P .

Lemma 6.4.4. *The following classes of morphism of schemes can be stratified into strongly equicohomological morphisms:*

- i) *proper morphisms,*
- ii) *affine morphisms of finite type.*

Proof. We observe that stratification is a purely topological notion and therefore we may assume that S is integral. Now, it suffices to prove that for the classes in question there exists an open subset $U \in S$ such that $f_U : X \times_S U \rightarrow U$ is strongly equicohomological. In both cases we apply the generic freeness (see [Sta17, Tag 051S]) to prove a more general statement that for any complex $\mathcal{E}^\bullet \in D_{\text{Coh}}^+(X)$ the higher direct images are generically free over S . Firstly, we deal with affine morphisms. In this case, the higher direct images of any complex $\mathcal{E}^\bullet \in D_{\text{Coh}}^+(X)$ are computed by taking the pushforwards of the cohomology groups $\mathcal{H}^j(\mathcal{E}^\bullet)$. Those are coherent sheaves on X and are therefore generically free over S . Secondly, we treat the case of proper morphism. Now, higher direct images $R^i f_* \mathcal{E}^\bullet$ are coherent S -modules and therefore are generically free. \square

We are now ready to state and prove the theorem.

Theorem 6.4.5. *Suppose $f : X \rightarrow S$ is a morphism which can be stratified into strongly equicohomological morphisms. Then, the set of geometric closed points $\bar{s} \in S$ such that the fibre $X_{\bar{s}} \stackrel{\text{def}}{=} X \times_S \text{Spec}(k(\bar{s}))$ lifts to $W_2(k(\bar{s}))$ is constructible.*

Proof. Observe that in order to prove constructibility we may stratify the scheme S into locally closed subsets. Therefore, by the assumption on existence of strongly equicohomological stratification, we may assume that $f : X \rightarrow S$ is strongly equicohomological with a smooth affine target. Then, the result follows from Lemma 6.4.3. \square

As a direct corollary of Lemma 6.4.4 and Theorem 6.4.5 we obtain:

Corollary 6.4.6. *For any proper or affine morphism $f : X \rightarrow S$ the set of geometric closed points $s \in S$ such that the fibre $X_{\bar{s}}$ lifts to $W_2(k(\bar{s}))$ is constructible.*

Chapter 7

Liftability of Frobenius morphism of singular schemes

We now proceed to the investigation of Frobenius liftability of schemes. In §7.1, we provide a characterization of obstruction classes for existence of a $W_2(k)$ -lifting compatible with Frobenius in case of singular schemes. Next, in §7.2 we present a computational criterion for Frobenius liftability of affine complete intersections. Then, we apply the criterion in the case of ordinary double points §7.3, and canonical surface singularities §7.4. Finally, in §7.5 we investigate Frobenius liftability of quotients, and then in §7.6 we relate the notion of Frobenius liftability with standard F -singularity types.

7.1 Some general results

We begin with a few general results concerning Frobenius liftability of singular schemes. Throughout, we freely use the functoriality properties of Exal-categories (cf. §3.2.1) and the assertions of Theorem 3.2.8.

Let X be a scheme defined over a perfect field of characteristic $p > 0$. First, we outline the relation between the set of liftings of the Frobenius morphism and the Witt scheme $(X, W_2(\mathcal{O}_X))$. Suppose that $\mathcal{O}_{X'}$ is a flat $W_2(k)$ -extension of \mathcal{O}_X . We consider the assignment:

$$\eta_X: W_2(\mathcal{O}_X) \rightarrow \mathcal{O}_{X'}, \quad (a_0, a_1) \mapsto \tilde{a}_0^p + p\tilde{a}_1, \text{ for some lifts of local section } a_0 \text{ and } a_1.$$

Using the relation

$$(\tilde{a}_0 + p\tilde{a}'_0)^p + p(\tilde{a}_1 + p\tilde{a}'_1) = \tilde{a}_0^p + p\tilde{a}_1 \pmod{p^2}$$

we see that η_X does not depend on the choices of lifts and determines a ring homomorphism. Moreover, if $\mathcal{O}_{X'}$ admits a lift of the Frobenius morphism $F': \mathcal{O}_{X'} \rightarrow \mathcal{O}_{X'}$ then there exists a homomorphism $\theta_{F'}: \mathcal{O}_{X'} \rightarrow W_2(\mathcal{O}_X)$ defined by the formula

$$\theta_{F'}: \mathcal{O}_{X'} \rightarrow W_2(\mathcal{O}_X), \quad \tilde{a} \mapsto \left(a, \frac{F'(\tilde{a}) - \tilde{a}^p}{p} \right),$$

where the division by p reflects the identification $p\mathcal{O}_{X'} \simeq \mathcal{O}_X$ given in Corollary 2.1.9. For every F' , the composition $\theta_{F'} \circ \eta_X$ is equal to the Witt vector Frobenius $\sigma_2: W_2(\mathcal{O}_X) \rightarrow W_2(\mathcal{O}_X)$ (cf. §2.1.1), and the composition $\eta_X \circ \theta_{F'}$ is equal to F' .

Lemma 7.1.1. *Let X be a scheme over a perfect field k of characteristic $p > 0$, and let $\mathcal{O}_{X'} \in \text{Exal}_{W_2(k)}(X, \mathcal{O}_X)$ be a flat $W_2(k)$ -extension of \mathcal{O}_X . Then $\mathcal{O}_{X'}$ admits a lift of Frobenius if and only if the push-forward*

$$(F^\#)_\times \mathcal{O}_{X'} \in \underline{\text{Exal}}_{W_2(k)}(X, F_* \mathcal{O}_X) \quad \text{defined as} \quad (\mathcal{O}_{X'} \oplus F_* \mathcal{O}_X) / \mathcal{O}_X$$

is isomorphic to $W_2(\mathcal{O}_X)$.

Proof. First, we prove that a lift of Frobenius $F': \mathcal{O}_{X'} \rightarrow \mathcal{O}_{X'}$ gives rise to an isomorphism $(F^\#)_\times \mathcal{O}_{X'} \simeq W_2(\mathcal{O}_X)$. Indeed, given F' there exists a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{p^\cdot} & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \downarrow F^\# & & \downarrow \theta_{F'} & & \parallel \\ 0 & \longrightarrow & F_* \mathcal{O}_X & \longrightarrow & W_2(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X \longrightarrow 0. \end{array}$$

Since $(F^\#)_\times \mathcal{O}_{X'}$ is the pushout of the left top corner of the left square of the diagram, this leads to a morphism $(F^\#)_\times \mathcal{O}_{X'} \rightarrow W_2(\mathcal{O}_X)$ of extension in $\underline{\text{Exal}}_{W_2(k)}(X, F_* \mathcal{O}_X)$ which is necessarily an isomorphism (cf. §3.2.1). Conversely, if $\iota: \mathcal{O}_{X'} \rightarrow F^\#_\times \mathcal{O}_{X'}$ is the natural inclusion, then an isomorphism $\phi: F^\#_\times \mathcal{O}_{X'} \rightarrow W_2(\mathcal{O}_X)$ induces a morphism $\eta_X \circ \phi \circ \iota: \mathcal{O}_{X'} \rightarrow \mathcal{O}_{X'}$ which provides a lift of the Frobenius. \square

In [MS87, Appendix, Proposition 1], the authors characterize the obstruction classes for existence of a mod p^2 lift compatible with Frobenius. We now generalize these results to the singular setting. Let $\mathcal{O}_X \rightarrow F_* \mathcal{O}_X \rightarrow B_X$ be the distinguished triangle induced by the morphism $F^\#: \mathcal{O}_X \rightarrow F_* \mathcal{O}_X$. Note that if X is normal and reduced B_X is in fact isomorphic to the sheaf B_X^1 of boundaries in the de Rham complex.

Theorem 7.1.2. *Let X be a reduced scheme defined over a perfect field k of characteristic p . Then there exists an obstruction class $\sigma_X^F \in \text{Ext}^1(L_{X/k}, B_X)$ whose vanishing is sufficient and necessary for existence of a $W_2(k)$ -lifting of X compatible with Frobenius.*

Proof. Let $f: X \rightarrow \text{Spec}(k)$ be the structural morphism of X/k . Using the distinguished triangle of cotangent complexes:

$$Lf^* L_{k/W_2(k)} \xrightarrow{df} L_{X/W_2(k)} \xrightarrow{r} L_{X/k} \longrightarrow Lf^* L_{k/W_2(k)}[1],$$

we obtain a diagram with exact rows and columns:

$$\begin{array}{ccccccc} \text{Ext}^1(Lf^* L_{k/W_2}, B_X) & \longrightarrow & \text{Ext}^1(L_{X/k}, B_X) & \xhookrightarrow{\text{or}} & \text{Ext}^1(L_{X/W_2(k)}, B_X) & \xrightarrow{\circ df} & \text{Ext}^1(Lf^* L_{k/W_2}, B_X) \\ \uparrow q \circ & & \uparrow q \circ & & \uparrow q \circ & & \uparrow q \circ \\ \text{Ext}^1(Lf^* L_{k/W_2}, F_* \mathcal{O}_X) & \longrightarrow & \text{Ext}^1(L_{X/k}, F_* \mathcal{O}_X) & \xrightarrow{\text{or}} & \text{Ext}^1(L_{X/W_2(k)}, F_* \mathcal{O}_X) & \xrightarrow{\circ df} & \text{Ext}^1(Lf^* L_{k/W_2}, F_* \mathcal{O}_X) \\ \uparrow F^\# \circ & & \uparrow F^\# \circ & & \uparrow F^\# \circ & & \uparrow F^\# \circ \\ \text{Ext}^1(Lf^* L_{k/W_2}, \mathcal{O}_X) & \longrightarrow & \text{Ext}^1(L_{X/k}, \mathcal{O}_X) & \xrightarrow{\text{or}} & \text{Ext}^1(L_{X/W_2(k)}, \mathcal{O}_X) & \xrightarrow{\circ df} & \text{Ext}^1(Lf^* L_{k/W_2}, \mathcal{O}_X). \end{array}$$

We claim that:

- (a) the image $q \circ [W_2(\mathcal{O}_X)] \in \text{Ext}^1(L_{X/W_2(k)}, B_X)$ of the extension class $[W_2(\mathcal{O}_X)] \in \text{Ext}^1(L_{X/W_2(k)}, F_* \mathcal{O}_X)$ uniquely lifts to an element:

$$\sigma_X^F \in \text{Ext}^1(L_{X/k}, B_X),$$

(b) the condition $\sigma_X^F = 0$ is sufficient and necessary for existence of a $W_2(k)$ -lifting of X admitting compatible lift of Frobenius.

To prove (a), we observe that $\text{Ext}^0(Lf^*L_{k/W_2(k)}, B_X) = 0$ (the complex $Lf^*L_{k/W_2(k)}$ is supported in degree ≤ -1 and B_X is a sheaf since X is reduced) and hence

$$\circ r: \text{Ext}^1(L_{X/k}, B_X) \rightarrow \text{Ext}^1(L_{X/W_2(k)}, B_X)$$

is injective, and therefore it suffices to prove that $(q \circ [W_2(\mathcal{O}_X)]) \circ df = 0$. However $(q \circ [W_2(\mathcal{O}_X)]) \circ df = q \circ ([W_2(\mathcal{O}_X)] \circ df)$ and hence we need to show that $[W_2(\mathcal{O}_X)] \circ df$ comes from an element in $\text{Ext}^1(Lf^*L_{k/W_2(k)}, F_*\mathcal{O}_X)$. By Remark 3.2.20 the element

$$[W_2(\mathcal{O}_X)] \circ df - F^\# \circ Lf^*[W_2(k)] \in \text{Ext}^1(Lf^*L_{k/W_2(k)}, F_*\mathcal{O}_X),$$

where $[W_2(k)] \in \text{Ext}^1(L_{k/W_2(k)}, k)$ corresponds to the extension $k \rightarrow W_2(k) \rightarrow k$, is the obstruction class to existence of a morphism of \mathbb{Z} -extensions filling the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & W_2(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & F_*k \simeq k & \longrightarrow & W_2(k) & \longrightarrow & k \longrightarrow 0. \end{array}$$

Such morphism clearly exists since the construction of Witt vectors of length two is functorial. Therefore the obstruction class is zero, which yields a unique element $\eta = Lf^*[W_2(k)]$ satisfying the relation $F^\# \circ \eta = [W_2(\mathcal{O}_X)] \circ df$. This proves existence of $\sigma_X^F \in \text{Ext}^1(L_{X/k}, B_X^1)$.

We now proceed to (b). First, we prove that condition $\sigma_X^F = 0$ is sufficient. Indeed, using exactness of the middle right column of the diagram, we see that under this assumption there exists an extension $[\mathcal{O}_{X'}] \in \text{Ext}^1(L_{X/W_2}, \mathcal{O}_X)$ such that $(F^\#)_\times \mathcal{O}_{X'} \simeq W_2(\mathcal{O}_X)$. Using above considerations, we see that the element $[\mathcal{O}_{X'}]$ satisfies the relation $F^\# \circ ([\mathcal{O}_{X'}] \circ df) = [W_2(\mathcal{O}_X)] \circ df = F^\# \circ Lf^*[W_2(k)]$, and therefore $[\mathcal{O}_{X'}] \circ df = Lf^*[W_2(k)]$ which implies that $\mathcal{O}_{X'}$ is a flat $W_2(k)$ -extension (cf. Remark 3.2.16). By Lemma 7.1.1 we see that such $\mathcal{O}_{X'}$ admits a Frobenius lifting. It is straightforward to see that Lemma 7.1.1 also proves that condition $\sigma_X^F = 0$ is necessary. \square

Remark 7.1.3. We claim that for any $W_2(k)$ -lifting X' of X the obstruction $\sigma_{X'}^F \in \text{Ext}^1(L_{X/k}, F_*\mathcal{O}_X)$ to lifting of the Frobenius morphism to X' maps to $\sigma_X^F \in \text{Ext}^1(L_{X/k}, B_X)$ under the natural homomorphism induced by $F_*\mathcal{O}_X \rightarrow B_X$. Indeed, let $\mathcal{O}_{X'} \in \underline{\text{Exal}}_{W_2(k)}(X, \mathcal{O}_X)$ be a flat $W_2(k)$ -extension of X . We have a diagram of sheaves of rings

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & W_2(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \parallel & & \downarrow \eta_X & & \downarrow F^\# \\ 0 & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & F_*\mathcal{O}_{X'} & \longrightarrow & F_*\mathcal{O}_X \longrightarrow 0. \end{array}$$

This induces an isomorphism between the push-forward $F_\times \mathcal{O}_{X'} \in \underline{\text{Exal}}_{W_2(k)}(X, F_*\mathcal{O}_X)$ and $W_2(\mathcal{O}_X)$, since $F_\times \mathcal{O}_{X'}$ is the fibred product of the bottom right corner of the right square of the diagram. Using compatibility conditions given in Remark 3.2.9, this translates to the equality:

$$dF \circ [\mathcal{O}_{X'}] = [W_2(\mathcal{O}_X)] \in \text{Ext}^1(L_{X/k}, F_*\mathcal{O}_X).$$

By Theorem 3.2.19 we see that the obstruction class $\sigma_{X'}^F$ for existence of a Frobenius lifting $F': \mathcal{O}_{X'} \rightarrow \mathcal{O}_{X'}$ equals $dF \circ [\mathcal{O}_{X'}] - F^\# \circ [\mathcal{O}_{X'}]$, and therefore $q \circ \sigma_{X'}^F = q \circ [W_2(\mathcal{O}_X)] = \sigma_X^F$ since $q \circ F^\# = 0$.

7.2 Computational criterion

In this section we address the problem of Frobenius liftability of affine schemes. In particular, we express the vanishing of obstruction classes for existence of a lifting of the Frobenius morphism of a given $W_2(k)$ -lifting as an explicit ideal containment problem. As a consequence, we derive a computational criterion for F -liftability in the case of affine complete intersection schemes (e.g., hypersurface singularities). Throughout this section we identify affine schemes with their associated algebras of global functions. Our considerations are based on §3.3, where we present standard results concerning naive cotangent complex.

7.2.1 General criterion

We work in the following setting. Let $R = k[x_1, \dots, x_n]$ and let $B = R/J$ for an ideal $J = (f_1, \dots, f_m)$. Let $R' = W_2[x_1, \dots, x_n]$ and let $J' = (f'_1, \dots, f'_m)$ be the ideal such that $B' = R'/J'$ is a flat $W_2(k)$ -lifting of B . Moreover, let $F': R' \rightarrow R'$ be the lifting of Frobenius morphism given by the formula:

$$\sum a_I \underline{x}^I \mapsto \sum \sigma(a_I) \underline{x}^{pI},$$

where $I = (i_1, \dots, i_n)$ is a multi-index, \underline{x}^I denotes the monomial $x_1^{i_1} \cdots x_n^{i_n}$ and σ_2 is the Witt vector Frobenius described in §2.1.1. Furthermore, let $w: R' \rightarrow R'$ be the base change homomorphism over $\sigma_2: W_2(k) \rightarrow W_2(k)$ defined by:

$$w\left(\sum a_I \underline{x}^I\right) = \sum \sigma_2(a_I) \underline{x}^I.$$

We plan to describe the obstruction class for liftability of Frobenius morphism of B to B' explicitly. For this purpose, we use the results of Theorem 3.3.3. For every flat $W_2(k)$ -module M we shall identify elements of pM with M/pM using Corollary 2.1.9 (by means of multiplication by p). We need the following:

Definition 7.2.1. Let $f' \in R'$. We define its *Frobenius residue* $P(f') \in R$ to be the unique polynomial $P(f')$ such that $F'(f') - f'^p = p \cdot P(f')$.

Using Frobenius residues, the obstruction classes to lifting of the Frobenius morphism can be described as follows.

Proposition 7.2.2. *Suppose B and B' be as above. Then, the Frobenius morphism of B lifts to B' if and only if there exists a sequence of elements $h_k \in P$, for $k = 1 \dots n$, such that for every $i \in \{1, \dots, m\}$ the Frobenius residue $P(f'_i)$ satisfies:*

$$P(f'_i) = \sum_{1 \leq k \leq n} \left(\frac{\partial f_i}{\partial x_k} \right)^p h_k \pmod{f_1, \dots, f_m}.$$

Proof. Using the results of Theorem 3.3.3, we see that the obstruction is an element of

$$\mathrm{Hom}_B(J/J^2, F_*B) / \mathrm{Hom}_B(\Omega_{P/k}^1 \otimes_P B, F_*B).$$

It is defined as the coset of a p -linear homomorphism (hence the lack of Frobenius push-forward below):

$$J/J^2 \rightarrow F_*B, \quad [f] \mapsto w(f')(x_1^{p'}, \dots, x_n^{p'}) \in pP'/J' \simeq B, \text{ for any lift } f' \text{ of } f$$

where $(x_1^p)', \dots, (x_n^p)'$ are preselected lifts of p -th powers of coordinate functions. A natural choice of such lifts is given by the polynomials x_1^p, \dots, x_n^p , and therefore the obstruction is determined by the p -linear homomorphism $\sigma: J/J^2 \rightarrow F_*B$:

$$J/J^2 \ni [f] \mapsto \sigma([f]) = w(f')(x_1^p, \dots, x_n^p) = F'(f').$$

Since σ is defined up to an element of J' we see that

$$\sigma([f]) = F'(f') - f'^p \pmod{J'}$$

which is equal to $pP(f') \in pP'/J' \simeq B$ by the definition of the Frobenius residue. Homomorphisms in $\text{Hom}_B(J/J^2, F_*B)$ lying in the image of $\text{Hom}_B(\Omega_{P/k}^1 \otimes_P B, F_*B)$ under the natural mapping are of the form:

$$J/J^2 \rightarrow F_*B, \quad [f] \mapsto \sum_{1 \leq k \leq n} \frac{\partial f}{\partial x_k} \cdot h_k = \sum_{1 \leq k \leq n} \left(\frac{\partial f}{\partial x_k} \right)^p h_k$$

for some elements $h_i \in F_*B$, for $i = 1, \dots, n$. The presence of p -th powers is the consequence of the fact that the B -structure on the target module is twisted by the Frobenius morphism. Ultimately, the obstruction for lifting Frobenius given by the coset of σ vanishes if and only if there exists a sequence of elements $h_i \in F_*B$, for $i = 1, \dots, n$, such that

$$\sigma(f_i) = P(f'_i) = \sum_{1 \leq k \leq n} \left(\frac{\partial f_i}{\partial x_k} \right)^p h_k,$$

for the generators f'_1, \dots, f'_m of the ideal J' . This proves our assertion. \square

7.2.2 Affine complete intersections

We now analyse how the above criterion works for affine complete intersection schemes. In this case, we derive a computationally feasible criterion to check whether they possess a $W_2(k)$ -lifting compatible with the Frobenius morphism. For this purpose we analyse how the Frobenius residues differ for various lifts of a given polynomial $f \in P$. Namely, we compute (again, using the identifications described in Lemma 2.1.9):

$$P(f' + pg) - P(f') = \frac{F'(f' + pg) - (f' + pg)^p}{p} - \frac{F'(f') - f'^p}{p} = \frac{F'(pg)}{p} = g^p.$$

This implies that the choice of a different lifting f' of f leads to the change of the Frobenius residue $P(f')$ some p -th power of an element $g \in R$.

Example 7.2.3. It turns out we have already seen an example of a polynomial $P(f')$. Namely, in Proposition 6.1.1 we defined the polynomial:

$$P_{f_{2n}} = \sum_{\substack{i_1 + \dots + i_n = p \\ i_1, \dots, i_n \neq p}} \frac{(p-1)!}{i_1! \dots i_n!} (x_1 x_2)^{i_1} \dots (x_{2n-1} x_{2n})^{i_n}.$$

This is exactly $-P(x_1 x_2 + \dots + x_{2n-1} x_{2n})$.

The above considerations lead to the following criterion:

Proposition 7.2.4. *The affine complete intersection algebra $A = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ is Frobenius liftable if and only if there exists a sequence of elements $h_k \in R$ and a sequence of elements $g_i \in R$ such that:*

$$P_{f_i} + g_i^p = \sum_{1 \leq k \leq n} \left(\frac{\partial f_i}{\partial x_k} \right)^p h_k \pmod{f_1, \dots, f_m}, \quad (7.1)$$

where P_{f_i} is a polynomial defined up to a p -th power of an element in R , and computed by the formula $P_{f_i} = P(f'_i)$ for some lifting f'_i of f_i .

Proof. By Lemma 2.1.8 we know that flat liftings of complete intersection affine schemes are given by the liftings of the associated regular sequences. Therefore, by Lemma 7.2.2 we need to find liftings $\{f'_i\}$ if $\{f_i\}$ such that there exist $\{h_k\}$

$$P(f'_i) = \sum_{1 \leq k \leq n} \left(\frac{\partial f'_i}{\partial x_k} \right)^p h_k \pmod{f_1, \dots, f_m},$$

But, as we mentioned, the polynomials $P(f'_i)$ for different liftings of f_i differ by arbitrary p -th powers. This finishes the proof. \square

Remark 7.2.5. Based on the description of obstruction classes given in Proposition 7.2.2 and Proposition 7.2.4 it is easy to see that a tuple of polynomials

$$(f'_1, \dots, f'_m, g_1, \dots, g_m, h_1, \dots, h_n)$$

satisfying:

$$P(f'_i) + g_i^p = \sum_{1 \leq k \leq n} \left(\frac{\partial f'_i}{\partial x_k} \right)^p h_k \pmod{f_1, \dots, f_m}$$

leads to an explicit lifting of Frobenius given by:

$$W_2(k)[x_1, \dots, x_n] / (f'_1 + pg_1, \dots, f'_m + pg_m) \xrightarrow{x_i \mapsto x_i^p + ph_i} W_2(k)[x_1, \dots, x_n] / (f'_1 + pg_1, \dots, f'_m + pg_m)$$

As a corollary of Proposition 7.2.4 we obtain the criterion for Frobenius liftability of hypersurface singularities:

Corollary 7.2.6. *Suppose $H_f = \text{Spec}(k[x_1, \dots, x_n]/(f))$ is a hypersurface in \mathbb{A}_k^n . Then H_f admits a lift to $W_2(k)$ compatible with Frobenius if and only if there exists an element g such that*

$$P_f + g^p \in \left(f, \left(\frac{\partial f}{\partial x_k} \right)^p, \dots, \left(\frac{\partial f}{\partial x_n} \right)^p \right).$$

Remark 7.2.7. The equality above might be treated as a relation in the Frobenius push-forward $F_*(k[x_1, \dots, x_n]/(f))$. From this perspective, the polynomial

$$P_f \in F_*(k[x_1, \dots, x_n]/(f))$$

(module structure by p -th powers) is supposed to belong to a sub-module

$$k[x_1, \dots, x_n]/(f) \cdot 1 + \left(\frac{\partial f}{\partial x_k}, \dots, \frac{\partial f}{\partial x_n} \right) \cdot F_*(k[x_1, \dots, x_n]/(f)).$$

Indeed, $g^p = g \cdot 1 \in F_*(k[x_1, \dots, x_n]/(f))$.

It turns out that the second form of the above criterion can be (quite) efficiently verified using `Macaulay2`. The source code is presented in §8.2.

Remark 7.2.8. The existence of Frobenius lifting depends on the choice of the $W_2(k)$ -lifting.

Indeed, Frobenius liftability of any lifting of $\text{Spec}(k[x_1, \dots, x_n]/(f))$ would mean that $P_f + g^p$ belongs to $(f, (\frac{\partial f}{\partial x_k})^p, \dots, (\frac{\partial f}{\partial x_n})^p)$ for any g and consequently $(f, (\frac{\partial f}{\partial x_k})^p, \dots, (\frac{\partial f}{\partial x_n})^p)$ contains all p -th powers. This is already not true for $f = x_1^3 + x_2^3 + x_3^3$ in characteristic $p \equiv 1 \pmod{3}$. In fact, a simple computation shows that

$$x_1^p \notin (x_1^3 + x_2^3 + x_3^3, x_1^{2p}, x_2^{2p}, x_3^{2p}).$$

Alternatively, one can evoke the theory of Serre and Tate to see that the canonical lifting of an ordinary elliptic curve is the only lifting of the curve to which the Frobenius morphism lifts.

7.3 Non-liftability of ordinary double points in any dimension

This section is devoted to answering the question posed in [Bha14, Remark 3.14], whether the ordinary double points, in particular cones over smooth projective quadrics, admit a lift to $W_2(k)$ compatible with Frobenius in arbitrary dimensions. The negative answer is given by:

Theorem 7.3.1 (Frobenius non-liftability of ordinary double points). *Let $n \geq 5$ be an integer and assume $p \geq 3$. Then the ordinary double points defined by the equation $f = x_1^2 + \dots + x_n^2 + Q$ for $Q \in k[x_1, \dots, x_n]_{\geq 3}$ do not admit a $W_2(k)$ -lifting compatible with Frobenius.*

Proof. For the proof, we need the following lemma combined with the Proposition 6.1.1.

Lemma 7.3.2 ([Lan08, Proposition 3.1] and [Lan10]). *Let k be a field of characteristic $p \geq 3$ and $0 \leq e < p$ be an integer. Then for any $d \leq N \cdot \frac{p-1}{2} - e$ we have*

$$\left((x_1^p, \dots, x_N^p) : \left(\sum_{i=1}^N x_i^2 \right)^e \right)_d = \left(x_1^p, \dots, x_N^p, \left(\sum_{i=1}^N x_i^2 \right)^{p-e} \right)_d$$

in $k[x_1, \dots, x_N]$.

By Corollary 7.2.6 we need to show that

$$P_f + g^p \notin \left(f, \left(\frac{\partial f}{\partial x_k} \right)^p, \dots, \left(\frac{\partial f}{\partial x_n} \right)^p \right)$$

for any $g \in k[x_1, \dots, x_n]$. Clearly, $\left(f, \left(\frac{\partial f}{\partial x_k} \right)^p, \dots, \left(\frac{\partial f}{\partial x_n} \right)^p \right) \subseteq (f, x_1^p, \dots, x_n^p)$ and therefore it suffices to prove that $P_f \notin (f, x_1^p, \dots, x_n^p)$. Assume the contrary, that is, there exists an $h \in k[x_1, \dots, x_n]$ such that:

$$P_f = P_{x_1^2 + \dots + x_n^2} + P' = h \cdot (x_1^2 + \dots + x_n^2 + Q) \pmod{x_1^p, \dots, x_n^p},$$

where P' is a polynomial in $k[x_1, \dots, x_n]_{>2p}$. The ring $k[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ admits a grading (induced from the polynomial ring). We may assume that the lowest degree

m of the polynomial h satisfies $m \geq 2p - 2$. Indeed, if $m < 2p - 2$ then by comparison of lowest degree terms and Lemma 7.3.2 for $N = 5$, $e = 1$ and $d = m$ we see that

$$h_{[m]} \in \left((x_1^p, \dots, x_n^p) : \left(\sum_{i=0}^n x_i^2 \right)_m \right) = \left(x_1^p, \dots, x_N^p, \left(\sum_{i=1}^N x_i^2 \right)^{p-1} \right)_m \subset (x_1^p, \dots, x_n^p)_m$$

and therefore $h - h_{[m]}$ is also an appropriate choice of h with the lowest degree at least $m + 1$. Consequently, by comparison of degree $2p$ terms we see that $P_{x_1^2 + \dots + x_n^2}$ belongs to the ideal $(x_1^2 + \dots + x_n^2, x_1^p, \dots, x_n^p)$. This gives a contradiction with the consideration after the relation (6.1) in the proof of Proposition 6.1.1. Namely f_n (resp. P_{f_n}) from Proposition 6.1.1 is the representation of the polynomial $x_1^2 + \dots + x_n^2$ (resp. $P_{x_1^2 + \dots + x_n^2}$) after standard linear transformation. More precisely, we have the following:

$$\begin{aligned} f_{2j}(x_1 + ix_2, x_1 - ix_2, x_3 + ix_4, x_3 - ix_4, \dots, x_{2j-1} + ix_{2j}, x_{2j-1} - ix_{2j}) &= x_1^2 + x_2^2 + \dots + x_{2j}^2 \\ f_{2j+1}(x_1 + ix_2, x_1 - ix_2, x_3 + ix_4, x_3 - ix_4, \dots, x_{2j-1} + ix_{2j}, x_{2j-1} - ix_{2j}, x_{2j+1}) &= x_1^2 + x_2^2 + \dots + x_{2j+1}^2. \end{aligned}$$

□

This answers the question asked in [Bha14, Remark 3.14] and in fact implies that Theorem 5.3.11 based on Frobenius liftability cannot be applied for ordinary double points. However, using alternative methods Bhatt proved that similar result also holds for proper schemes with ordinary points.

7.4 Frobenius liftability of canonical singularities

In this section, we present two different approaches to proving that canonical singularities of surfaces are F -liftable. The first one is a direct computation using the criterion given in Corollary 7.2.6. The second gives a slightly weaker result in the case of tame quotient singularities and exploits the structure of the quotient together with functoriality of obstructions for lifting Frobenius morphism. We note that in characteristic ≥ 7 the canonical surface singularities over algebraically closed field are classified by the dual graphs of the minimal resolution, which in turn correspond to Dynkin diagrams of type A, D and E. Therefore, we shall use the terms canonical surface singularities and ADE singularities interchangeably.

7.4.1 Direct computational approach

Firstly, we approach the F -liftability of ADE singularities by a direct computation. By [Art77] under the assumption $p \geq 7$ the ADE singularities are locally analytically equivalent to the germs of the form described in Table 7.1

Table 7.1: Models of canonical singularities in char. $p \geq 7$

Type	Equation
smooth	-
A_{n-1}	$x^n + y^2 + z^2 = 0$
D_{n+2}	$x^2 + y^2 z + z^{n+1} = 0$
E_6	$x^2 + y^3 + z^4 = 0$
E_7	$x^2 + y^3 + yz^3 = 0$
E_8	$x^2 + y^3 + z^5 = 0$

Equipped with this classification we can approach the proof of the following:

Theorem 7.4.1. *For any algebraically closed field k of characteristic $p \geq 7$ any affine scheme X with canonical surface singularities is Frobenius liftable.*

Proof. Firstly, we observe that by Lemma 3.5.2 it suffices to prove that for any $x \in X$ the germ $\text{Spec}(\mathcal{O}_{X,x})$ is Frobenius liftable. By Artin's approximation theorem (see [Art69, Corollary 2.6]) and the analytical classification of canonical singularities there exists a diagram of schemes:

$$\begin{array}{ccc} & (U, u) & \\ \text{étale} \swarrow & & \searrow \text{étale} \\ \text{Spec}(\mathcal{O}_{X,x}) & & \text{Spec}(\mathcal{O}_{S_{\text{ADE},s}}), \end{array}$$

where $\mathcal{O}_{S_{\text{ADE},s}}$ is the local ring at 0 of one of the affine models given in Table 7.1. Therefore, by Lemma 3.5.1 and Lemma 3.5.2, to prove Frobenius liftability of $\text{Spec}(\mathcal{O}_{X,x})$ it suffices to show that the affine models given in Table 7.1 are F -liftable.

The case of A_{n-1} singularities simply follows from Example 5.1.8 as A_{n-1} singularities are toric. Consequently, we treat the case of D_{n+2} singularities. The affine model of D_{n+2} is given by the equation $x^2 + y^2z + z^{n+1} = 0$ and therefore by Corollary 7.2.6 we are left to show that:

$$\sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{\binom{p}{i,j,k}}{p} x^{2i} (y^2z)^j z^{(n+1)k} \in (x^2 + y^2z + z^{n+1}, x^p, y^p z^p, y^{2p} + (n+1)^p z^{np}).$$

By substituting $u = x^2 = -y^2z - z^{n+1}$ we see that it suffices to prove that:

$$\sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{\binom{p}{i,j,k}}{p} (-1)^i (y^2z + z^{n+1})^i (y^2z)^j z^{(n+1)k} \in ((y^2z + z^{n+1})^{\lceil p/2 \rceil}, y^p z^p, y^{2p} + (n+1)^p z^{np}).$$

The left hand side of the above relation is a sum of elements of the form $(y^2z)^a \cdot z^{(n+1)b}$ for $a + b = p$ so it is enough to show that:

$$(y^2z)^a z^{(n+1)b} \in ((y^2z + z^{n+1})^{\lceil p/2 \rceil}, y^p z^p, y^{2p} + (n+1)^p z^{np}),$$

for any $a + b = p$. The claim is clear for every $a \geq \lceil p/2 \rceil$. Let us therefore assume that for some $a \leq \lfloor p/2 \rfloor$ we have

$$(y^2z)^a z^{(n+1)(p-a)} \notin ((y^2z + z^{n+1})^{\lceil p/2 \rceil}, y^p z^p, y^{2p} + (n+1)^p z^{np}).$$

Let us consider maximal such a . As $p - a \geq \lceil p/2 \rceil$ we can write

$$\begin{aligned} (y^2z)^a \cdot z^{(n+1)(p-a)} &= (y^2z)^a z^{(n+1)\lceil p/2 \rceil} z^{(n+1)(p-a-\lceil p/2 \rceil)} \\ &= (y^2z)^a z^{(n+1)(p-a-\lceil p/2 \rceil)} \cdot \sum_{1 \leq i \leq p} \binom{\lceil p/2 \rceil}{i} (y^2z)^i z^{(n+1)(\lceil p/2 \rceil - i)} \\ &= \sum_{1 \leq i \leq p} \binom{\lceil p/2 \rceil}{i} (y^2z)^{a+i} z^{(n+1)(p-a-i)}. \end{aligned}$$

This gives a contradiction with the definition of a .

We now proceed to the case of E_7 . We shall prove that

$$\sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{\binom{p}{i,j,k}}{p} x^{2i} y^{3j} (yz^3)^k \in (x^2 + y^3 + yz^3, x^p, 3y^{2p} + z^{3p}, y^p z^{2p}).$$

By the same trick, it suffices to prove that for any $a + b = p$ we have

$$(yz^3)^a y^{3b} \in ((y^3 + yz^3)^{\lceil p/2 \rceil}, 3y^{2p} + z^{3p}, y^p z^{2p}),$$

which follows by similar arguments.

We finish with case of E_{2m} singularities given by the affine equation $x^2 + y^3 + z^{m+1}$ for $m = 3, 4$. By Corollary 7.2.6 to prove the existence of $W_2(k)$ -lifting compatible with the Frobenius morphism it suffices to prove that

$$\sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{\binom{p}{i,j,k}}{p} x^{2i} y^{3j} z^{(m+1)k} \in (x^2 + y^3 + z^{m+1}, x^p, y^{2p}, z^{mp}).$$

After substituting $u = x^2$, $v = y^3$, $w = z^{m+1}$ and $s = u + v + w$ it suffices to show that:

$$\sum_{\substack{i+j+k=p \\ i,j,k \neq p}} \frac{\binom{p}{i,j,k}}{p} u^i v^j (s-u-v)^k \in (s, u^{\lceil p/2 \rceil}, v^{\lceil 2p/3 \rceil}, (s-u-v)^{\lceil \frac{mp}{m+1} \rceil}) = (s, u^{\lceil p/2 \rceil}, v^{\lceil 2p/3 \rceil}, (u+v)^{\lceil \frac{mp}{m+1} \rceil}).$$

For this purpose, we prove that any monomial $u^i v^j \in k[u, v]$ for $i + j = p$ appearing on the left hand side belongs to $(u^{\lceil p/2 \rceil}, v^{\lceil 2p/3 \rceil}, (u+v)^{\lceil \frac{mp}{m+1} \rceil})$. This is equivalent to the surjectivity of the linear mapping defined by:

$$\begin{aligned} k[u, v]_{\lfloor p/2 \rfloor} \times k[u, v]_{\lfloor p/3 \rfloor} \times k[u, v]_{\lfloor \frac{p}{m+1} \rfloor} &\rightarrow k[u, v]_p \\ (f, g, h) &\mapsto f \cdot u^{\lceil p/2 \rceil} + g \cdot v^{\lceil 2p/3 \rceil} + h \cdot (u+v)^{\lceil mp/(m+1) \rceil}. \end{aligned}$$

After writing this in the basis given by monomials $u^k v^j$ ordered lexicographically, to show surjectivity it is sufficient to show that the determinant of the matrix $\left[\binom{\lceil mp/(m+1) \rceil}{s+i-j} \right]_{1 \leq i, j \leq n}$, for $n = p - 1 - \lfloor p/2 \rfloor - \lfloor p/3 \rfloor$ is non-zero. Using [Kra05, Formula (2.1)], the determinant is equal to:

$$\det \left[\binom{\lceil \frac{mp}{m+1} \rceil}{s+i-j} \right]_{i,j \leq n} = \prod_{i=1}^n \frac{(\lceil \frac{mp}{m+1} \rceil + i - 1)!(i - 1)!}{(s + i - 1)!(\lceil \frac{mp}{m+1} \rceil - s + i - 1)!},$$

and therefore we are done by the inequality $\lceil \frac{mp}{m+1} \rceil + n - 1 < p$. \square

7.5 Quotient singularity approach

Here, we approach the F -liftability of canonical singularities of surfaces using a different method based on functoriality of obstructions. We begin with a general result concerning F -liftability of quotients of smooth schemes by finite groups, for which the action can be lifted to characteristic 0 (in fact, lifting mod p^2 would be sufficient). Subsequently, we investigate to which extent our method can be applied in the case of canonical surfaces singularities. Our analysis is based on the results of [LS14, Section 4] describing whether ADE singularities admit a structure of a quotient singularity.

7.5.1 Actions of finite groups

We precede our actual considerations by a few essential and standard remarks concerning base change for invariants of actions of finite groups. The proofs are based on the existence of so-called Reynolds operator.

Lemma 7.5.1. *Let G be a finite group and let R be a commutative ring such that $|G|$ is invertible in R . Then, any $R[G]$ -module M projective as an R -module is projective as an $R[G]$ -module.*

Proof. In order to prove that M is projective we need show that for any diagram of $R[G]$ -modules:

$$\begin{array}{ccc} & M & \\ & \downarrow f & \\ P & \xrightarrow{p} & N \longrightarrow 0 \end{array}$$

there exists an $R[G]$ -linear homomorphism $s : M \rightarrow P$ such that $f = p \circ s$. By the assumptions we obtain an R -linear homomorphism $\sigma : M \rightarrow P$ such that $f = p \circ \sigma$. Consequently, by a direct computation we obtain that a homomorphism defined by the formula (i.e., the Reynolds operator):

$$s(m) = \frac{1}{|G|} \sum_{g \in G} g\sigma(g^{-1}m)$$

is $R[G]$ -linear and satisfies $f = p \circ s$. □

We now apply the above result to prove:

Lemma 7.5.2. *Let G be a finite group and let R be a commutative ring such that $|G|$ is invertible in R . Let M be a projective R -module together with an action of G . Then, for any homomorphism $R \rightarrow S$ the natural mapping $M^G \otimes_R S \rightarrow (M \otimes_R S)^G$ is an isomorphism.*

Proof. Firstly, by Lemma 7.5.1 we observe that M is a projective $R[G]$ -module. Therefore, there exists an $R[G]$ -module N such that $M \oplus N \simeq R[G]^I$ for some set of indices I . Consequently, we obtain the following diagram:

$$\begin{array}{ccc} (M \oplus N)^G \otimes_R S & \xrightarrow{\simeq} & M^G \otimes_R S \oplus N^G \otimes_R S \\ \downarrow \simeq & & \downarrow \\ (R[G]^I)^G \otimes_R S & \longrightarrow & (M \otimes_R S)^G \oplus (N \otimes_R S)^G \simeq (R[G] \otimes_R S)^G, \end{array}$$

which reduces our claim to the case of free modules. We finish by the following simple sequence of identifications:

$$(R[G]^I)^G \otimes_R S \rightarrow R^I \otimes_R S \rightarrow S^I \rightarrow (S[G]^I)^G \rightarrow (R[G]^I \otimes_R S)^G.$$

□

As a corollary we obtain:

Corollary 7.5.3. *Let G be a finite group and let R be a ring such that $|G|$ is invertible in R . Then, for every G -action over $\mathrm{Spec}(R)$ on $\mathbb{A}_{\mathrm{Spec}(R)}^n$ the quotient $\mathbb{A}_{\mathrm{Spec}(R)}^n/G$ is flat over R . Moreover, for every ring homomorphism $R \rightarrow S$ the natural map:*

$$\mathbb{A}_{\mathrm{Spec}(S)}^n/G \rightarrow \left(\mathbb{A}_{\mathrm{Spec}(R)}^n/G \right) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S).$$

is an isomorphism.

Proof. Using the same idea as in Lemma 7.5.1 we prove that the ring of invariants $R[x_1, \dots, x_n]^G$ is a direct sum of a flat R -module $R[x_1, \dots, x_n]$. This implies that $\mathbb{A}_{\mathrm{Spec}(R)}^n/G = \mathrm{Spec}(R[x_1, \dots, x_n]^G)$ is flat over R . The base change property is a direct consequence of Lemma 7.5.2. \square

7.5.2 Frobenius liftability of quotient singularities

We are now ready to approach the following result concerning F -liftability of quotient singularities.

Lemma 7.5.4 (Frobenius liftability of mod p reductions of quotients). *Let G be a finite group and let T be a spectrum of a finitely generated \mathbb{Z} -algebra which is étale over $\mathrm{Spec}(\mathbb{Z}[1/|G|])$. Suppose, G acts on \mathbb{A}_T^n relatively to T . Then, for every perfect field k and every k -point $t \in T(k)$ the scheme $\mathbb{A}_t^n/G \simeq (\mathbb{A}_T^n/G)_t$ is $W_2(k)$ -liftable compatibly with the Frobenius morphism.*

Proof. Let $t \in T(k)$ be k -point of T . By the formal lifting property of the étale map $T \rightarrow \mathrm{Spec}(\mathbb{Z})$, any point $t \in T(k)$ can be lifted to a point $\tilde{t} \in T(W_2(k))$. Consequently, by the flatness of $\mathbb{A}_T^n/G \rightarrow T$ (see Corollary 7.5.3) we obtain a $W_2(k)$ -lifting $(\mathbb{A}_T^n/G)_{\tilde{t}}$ of $(\mathbb{A}_T^n/G)_t$ fitting into a commutative diagram with cartesian squares:

$$\begin{array}{ccccc} \mathbb{A}_t^n & \xrightarrow{\quad} & \mathbb{A}_{\tilde{t}}^n & & \\ \pi_t \searrow & & \downarrow & \searrow \pi_{\tilde{t}} & \\ (\mathbb{A}_T^n/G)_t \simeq \mathbb{A}_t^n/G & \xrightarrow{\quad} & & & (\mathbb{A}_T^n/G)_{\tilde{t}} \\ \swarrow & & \downarrow & \swarrow & \\ \mathrm{Spec}(k) & \xrightarrow{\quad} & \mathrm{Spec}(W_2(k)) & & \end{array}$$

By Corollary 7.5.3 the scheme $(\mathbb{A}_T^n/G)_t$ is isomorphic to \mathbb{A}_t^n/G and π_t is a quotient map $\mathbb{A}_t^n \rightarrow \mathbb{A}_t^n/G$. Hence by the assumptions on T the degree $\deg(\pi_t)$ (equal to the length of a generic orbit) is coprime to the characteristic of k and therefore there exists a splitting operator of $\pi_t^\#$, which consequently allows us to apply Lemma 3.7.2 (for the upper square with mappings π_t and $\pi_{W_2(k)}$) to obtain our claim. More precisely, the Frobenius lifting of $\mathbb{A}_{W_2(k)}^n$ descends to $(\mathbb{A}_T^n/G)_{W_2(k)}$, which is a $W_2(k)$ -lifting of $(\mathbb{A}_T^n/G)_{W_2(k)}$. \square

In order to apply Lemma 7.5.4 to the case of canonical singularities of surfaces we need some results from [LS14]. First, let us recall the definition:

Definition 7.5.5. A scheme over a field k has *linearly reductive quotient singularities* (resp. *tame quotient singularities*) if it is étale equivalent to a quotient of a smooth k -scheme by a finite linearly reductive group scheme (resp. finite étale group scheme of order prime to the characteristic of k).

It is proven in [LS14, Prop. 4.2] that except for a few characteristics all ADE singularities fit into the above definition. Moreover, from the proof we can infer the following result concerning characteristic 0 liftability of group actions leading to ADE singularities:

Proposition 7.5.6. *For any singularity type $\tau \in \{A_{n-1}, D_{n+2}, E_6, E_7, E_8\}$ there exists a finite flat generically étale group scheme $\Lambda_\tau / \operatorname{Spec}(\mathbb{Z})$ such that for every field k the type τ singularity over k arises as the quotient $\mathbb{A}_k^2 / \Lambda_k$. The group scheme Λ is étale precisely over an open subset of $\operatorname{Spec}(\mathbb{Z})$ where the corresponding group scheme quotient is tame (in the sense of the above definition).*

Example 7.5.7. For example A_{n-1} singularity over a field k of arbitrary characteristic arises as the quotient of $\mathbb{A}_k^2 / \Lambda_k$ by the natural action of the group scheme $\mu_{n,k} \simeq \Lambda_k = \Lambda \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(k)$ for $\Lambda = \mu_{n,\mathbb{Z}}$. In this case, the scheme Λ/\mathbb{Z} is étale over $\operatorname{Spec}(\mathbb{Z}[1/n])$.

Finally, the details of the behaviour of canonical surface singularities with respect to characteristic of the field k can be summarised by the following table from [LS14, Section 4].

Table 7.2: Classification of canonical singularities of surfaces		
	Linearly reductive quotient singularity	Tame quotient singularity
A_{n-1}	every p	$p \nmid n$
D_{n+2}	$p \geq 3$	$p \geq 3, p \nmid n$
E_6	$p \geq 5$	$p \geq 5$
E_7	$p \geq 5$	$p \geq 5$
E_8	$p \geq 7$	$p \geq 7$

Note that the last column indicates the open subset where the respective group scheme Λ is étale. We are now ready to give an alternative proof of F -liftability of canonical singularities of surfaces in the case they are tame quotient singularities.

Theorem 7.5.8 (Tame version of Theorem 7.4.1). *Suppose X is an affine surface over an algebraically closed field k such that its singularities are tame quotient canonical surface singularities. Then, X is Frobenius liftable.*

Proof. Again by Lemma 3.5.1 and Lemma 3.5.2 it suffices to address the case of tame quotients. By Proposition 7.5.6 and the tameness assumption there exists an open subset $U = \operatorname{Spec}(\mathbb{Z}[1/N]) \subset \operatorname{Spec}(\mathbb{Z})$ and an étale group scheme Λ/U of rank coprime to N such that $X \simeq \mathbb{A}_u^2 / \Lambda_u$ for a certain point $u \in U(k)$. Every finite étale group scheme is étale locally constant and therefore there exists an étale morphism $T \rightarrow U$ such that $\Lambda_T \stackrel{\text{def}}{=} \Lambda \times_U T$ is isomorphic to a constant group scheme associated to a finite group G . As k is algebraically closed, the point $u \in U(k)$ lifts to $t \in T(k)$. By Lemma 7.5.4 applied for G acting on \mathbb{A}_T^2 the fibre $\mathbb{A}_u^2 / \Lambda_u \simeq \mathbb{A}_t^2 / G \simeq (\mathbb{A}_T^2 / G)_t$ is F -liftable. This finishes the proof. \square

Example 7.5.9. A D_{n+2} singularity arises as the quotient of \mathbb{A}_k^2 by the action of the binary dihedral group scheme BD_n of rank $4n$ induced by the natural operation of the matrices:

$$\begin{bmatrix} \xi_{2n} & 0 \\ 0 & \xi_{2n}^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

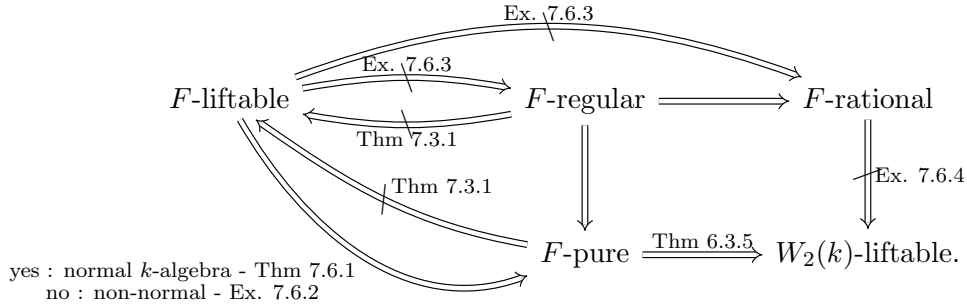
where ξ_{2n} denotes the $2n$ -th primitive root of unity. This means that the model of D_{n+2} is tame over $\operatorname{Spec}(\mathbb{Z}[1/2n])$ and the associated étale group scheme is trivialized by the

covering induced by $\mathbb{Z} \rightarrow \mathbb{Z}[1/2n][\xi_{2n}]$. For more details, the reader is referred to [LS14, Section 4].

7.6 $W_2(k)$ -liftability and F -liftability compared to standard F -singularities

The following section contains the comparison of the classical F -singularity types, i.e., F -regularity, F -purity and F -rationality with the notions of $W_2(k)$ -liftability and Frobenius liftability.

We are now ready to describe our results. They can be summarized by the following diagram indicating possible implications and referring to counterexamples in case they do not hold. The existence of counterexamples is denoted by a strike-through arrow. Moreover, the implication F -liftable $\Rightarrow F$ -pure holds under special assumptions described in the comments below the corresponding arrow.



Firstly, we show that an F -liftable normal scheme over k is Frobenius split.

Theorem 7.6.1. *Let X/k be normal scheme locally of finite type over a perfect field k , such that there exists a lifting $\tilde{X}/W_2(k)$ together with a lifting \tilde{F} of Frobenius morphism F . Then, X is canonically Frobenius split.*

Proof. The following is a simple extension of a proof given in [MS87] covering the smooth case. Let n be the dimension of X and let U be the smooth locus of X . In case of smooth schemes liftability of Frobenius is equivalent to the existence of a splitting $\xi : \Omega_U^1 \rightarrow Z_U^1$ of the Cartier mapping C (see [MS87]). After taking the n -th exterior power of ξ , we obtain a mapping $\wedge^n \xi : \Omega_U^n \rightarrow Z_U^n \simeq F_* \Omega_U^n$ which splits the sequence:

$$0 \longrightarrow B_U^n \longrightarrow Z_U^n \simeq F_* \Omega_U^n \longrightarrow \Omega_U^n \longrightarrow 0,$$

dual to the sequence

$$0 \longrightarrow \mathcal{O}_U \longrightarrow F_* \mathcal{O}_U \longrightarrow B_U^1 \longrightarrow 0,$$

by applying $\mathcal{H}om_{\mathcal{O}_U}(-, \Omega_U^n)$. Therefore, U is Frobenius split. By normality of X , we know that for any reflexive sheaf, in particular $\mathcal{H}om$ sheaves, one may extend sections along codimension 2 subsets, and therefore the splitting extends to a splitting of X . \square

We shall now present an example which violates the hypothesis of Theorem 7.6.1 in case of non-normal schemes.

Example 7.6.2. The scheme $\{x_1x_2(x_1 + x_2) = 0\} \subset \mathbb{A}_k^2$ is Frobenius liftable but not F -pure.

Proof. To prove F -purity we apply Fedder's criterion given in Lemma 4.5.7 and a fact that:

$$(x_1x_2(x_1 + x_2))^{p-1} \in (x_1^p, x_2^p).$$

Frobenius liftability follows by observing that the canonical lifting of Frobenius of $W_2[x_1, x_2]$ given by $x_i \mapsto x_i^p$ preserves the ideal $x_1x_2(x_1 + x_2)$. \square

Example 7.6.3. The cone $C = \{x^3 + y^3 + z^3 = 0\} \subset \mathbb{A}_k^3$ over the Fermat cubic in characteristic $p \equiv 1 \pmod{3}$ is F -liftable but neither strongly F -regular nor F -rational.

Proof. Firstly, F -liftability follows from Corollary 7.2.6 by a direct computation. We verify that C is strongly F -regularity by means of Lemma 4.5.7. Indeed, we observe that for any $s \in \mathfrak{m} = (x_1, x_2, x_3)$ we have $sf^{p^e-1} \in (x_1, x_2, x_3)^{[p^e]}$ and consequently the cone is not strongly F -regular at \mathfrak{m} . Finally, the proof of the fact that the cone C is not F -rational is contained in [TW15, Example 2.6]. \square

Example 7.6.4. There exists an F -rational scheme which is not $W_2(k)$ -liftable.

Proof. Let X be a smooth scheme embedded into \mathbb{P}_k^n which is not $W_2(k)$ -liftable (e.g., a smooth non-liftable surface from [Ray78]). By [LS14, Theorem 2.3] we see that $Y \stackrel{\text{def}}{=} \text{Bl}_X(\mathbb{P}_k^n)$ does not lift to $W_2(k)$, either. We claim that a cone over sufficiently ample projective embedding of Y satisfies our requirements, i.e., is F -rational and does not lift to $W_2(k)$.

Indeed, using [CR11] and the Leray spectral sequence we see that $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$, and therefore we may apply Proposition 2.5.2 and Proposition 3.6.19 to obtain a projective embedding $Y \subset \mathbb{P}^N$ such that the cohomology groups $H^i(Y, \mathcal{O}_Y(k))$ vanish for $i > 0$ and $k > 0$, the cone $C_Y \stackrel{\text{def}}{=} \text{Cone}_{Y, \mathbb{P}^N}$ is Cohen-Macaulay and does not lift to $W_2(k)$. Consequently, in order to apply Lemma 4.5.21 (which might require additional Veronese embedding) we are left to show that the positive degree part $[H_{\mathfrak{m}}^{n+1}(C_Y, \mathcal{O}_{C_Y})]_{\geq 0}$ of the graded module $H_{\mathfrak{m}}^{n+1}(C_Y, \mathcal{O}_{C_Y})$ is zero. This follows from the identification $[H_{\mathfrak{m}}^{n+1}(C_Y, \mathcal{O}_{C_Y})]_k = H^n(Y, \mathcal{O}_Y(k)) = 0$ given in Proposition 2.5.2 \square

Chapter 8

Appendix A

In the appendix, we present the source code and comment on our Macaulay2 scripts for checking $W_2(k)$ and Frobenius liftability. The current versions of scripts are available at <http://www.mimuw.edu.pl/~mez/Macaulay2>.

8.1 Checking mod p^2 liftability

The following code directly computes the obstruction class described in [Har10, Theorem 10.1].

```
checkExistence=method(TypicalValue=>Matrix,Options=>{SourceRing=>null})
checkExistence := F -> (
    S = ring F; p = char S;
    if p == 0 then error "Witt vector liftability makes sense only in char p>0";
    R = ZZ[gens S]; B = S/image(F);
    liftF = sub(F,R); -- we lift to ZZ[gens S]
    Q = gens ker F;
    stdio << "Source and target: " << source Q << " " << target Q << endl;
    liftQ = sub(Q,R);
    I = liftF*liftQ;
    val = sub(sub(matrix{{1/p_R}}*promote(I,frac(R)),R),S); -- we divide by p
    val' = sub(transpose(val),B); -- ' means reduced to B
    Q' = sub(transpose(Q),B);
    result = (image(val') + image(Q') == image(Q'));
    use S;
    result
)

p = 5
S = ZZ/p[x_1..x_6]
F = matrix{apply(gens S,f -> f^p)} | matrix{{x_1*x_2 + x_3*x_4 + x_5*x_6}}
print F;
time (flag = checkExistence(F));
print flag
```

8.2 Checking Frobenius liftability

In this script, using the idea expressed in Remark 7.2.7, we check whether a hypersurface ring is Frobenius liftable.

```
load("PushForward.m2")
p := 11
S := ZZ/p[x,y,z]
lF := x^2 + y^3 + z^7 + x*y*z
S' := ZZ[gens S]
```

```

liftF = sub(lF,S') -- we lift to ZZ[gens S]
liftFp = sub(lF^p,S') -- lift of p-th power
liftdP = (liftFp - liftF^p)//p

stdio << endl << "Char = " << p << endl << "Polynomial: " << lF << endl

-- SIMPLE

dPP = sub(liftdP,S)
I = ideal(matrix{apply(flatten(entries(jacobian(matrix{{lF}}))),u -> u^p)} | matrix{{lF}})
stdio << I << endl;
stdio << "Is Frobenius-liftable? [SIMPLE]: " << isSubset(ideal(dPP),I) << endl

-- COMPLEX

R := S/ideal(lF)
use R
dP = sub(liftdP,R)
Frob := map(R,R,matrix{apply(gens R,u -> u^p)})
(M,G,transform) := pushFwd(Frob)
lI := jacobian(matrix{{lF}})
I := ideal(sub(lI,R))
Pf := map(M,R^1,transform(dP))
Unit := map(M,R^1,transform(1))
flag := isSubset(image Pf,(image Unit) + I*M)
stdio << "Is Frobenius-liftable? [COMPLEX]: " << flag << endl

```

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